VOLUME INEQUALITIES FOR SUBSPACES OF L_p

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Abstract

A direct approach is used to establish both Ball and Barthe's reverse isoperimetric inequalities for the unit balls of subspaces of L_p. This approach has the advantage that it completely settles all the open uniqueness questions for these inequalities.

Affine isoperimetric inequalities generally have ellipsoids as extremals. The so called *reverse* affine isoperimetric inequalities usually have simplices — or in the symmetric case cubes and their polars — as their extremals.

Symmetrization techniques, developed and promoted by Steiner well over a century ago, have been used to establish a variety of powerful affine isoperimetric inequalities. The reverse inequalities would turn out to be much harder to establish. They appeared to require some sort of *anti*symmetrization technique.

By 1990, only one significant reverse inequality had been established in dimensions greater than two: the Rogers-Shephard *difference-body inequality* (see e.g., Schneider [**42**]). Unfortunately, the techniques employed by Rogers and Shephard could not be adapted to establish any of the other conjectured reverse inequalities. A breakthrough occurred in 1990 when Keith Ball connected John's theorem characterizing the largest ellipsoid contained in a convex body (the *John ellipsoid*) with the Brascamp-Lieb inequality. The Brascamp-Lieb inequality had been developed to solve the best-constant problem for Young's convolution inequality (see the excellent recent survey of Gardner [**11**]). Ball discovered a gorgeous reformulation of the Brascamp-Lieb inequality that seemed tailor-made to exploit the John ellipsoid. Ball's *normalized* Brascamp-Lieb inequality has had a profound impact on convex geometric analysis (see, e.g., Ball [**1**, **2**, **3**], Bastero and Romance [**5**], Giannopoulos and Papadimitrakis [**13**], Giannopoulos, Milman, and Rudelson [**12**], Giannopoulos, Perissinaki, and Tsolomitis [**14**], Schechtman and Schmuckenschläger [**40**], Schmuckenschläger [**41**]).

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The problem of characterizing the extremals of the reverse inequalities remained open. The reason for this is that not only were the equality conditions of the Brascamp-Lieb inequality not known, but they were known to be complicated. The breakthrough here was achieved by Barthe [**4**]. Barthe not only established the equality conditions for the Brascamp-Lieb inequality but he discovered an amazing new approach to establishing the Brascamp-Lieb inequality. Among other things, this allowed Barthe to settle the uniqueness question for those reverse inequalities established using the Brascamp-Lieb inequality where the generating measure is discrete.

Basic questions (discussed below) in the L_p *Brunn-Minkowski theory* (see, e.g., [**7**], [**8**], [**9**], [**15**], [**16**], [**17**], [**22**], [**23**], [**24**], [**25**], [**26**], [**27**], [**28**], [**30**], [**29**], [**31**], [**32**], [**33**], [**34**], [**35**], [**39**], [**43**], [**44**], [**45**], [**46**], [**48**]) motivated the search for a direct approach to establishing these reverse inequalities. A simple approach, which does not use the Brascamp-Lieb inequality, is one of the aims of this paper. This approach allows us to answer all the surrounding uniqueness questions that had remained open.

Throughout, a Borel measure on unit sphere S^{n-1} of Euclidean n-space $\mathbb{R}^n$ is to be understood to mean a nonnegative finite Borel measure on S^{n-1}. A Borel measure μ on S^{n-1} is said to be *isotropic* provided

$$\int_{S^{n-1}} v \otimes v \, d\mu(v) = I,$$

where I denotes the identity operator on $\mathbb{R}^n$, and $v \otimes v$ is the rank 1 linear operator on $\mathbb{R}^n$ that takes x to $(x \cdot v)v$, where $x \cdot v$ denotes the standard inner product of x and v in $\mathbb{R}^n$. A measure on S^{n-1} is said to be *even* if it assumes the same value on antipodal sets. Each even isotropic Borel measure μ on S^{n-1} determines an n-dimensional subspace of L_p whose unit ball we denote by $Z_p^* = Z_p^*(\mu)$; to be specific, the n-dimensional subspace of $L_p = L_p(S^{n-1})$ may be taken to be $\mathbb{R}^n$ with a norm defined, for each $x \in \mathbb{R}^n$, by

$$\|x\|_{Z_p^*} = \Big[\int_{S^{n-1}} |x \cdot v|^p d\mu(v)\Big]^{1/p}.$$

Conversely, a theorem of Lewis [**20**] shows that each n-dimensional subspace of L_p is isometric to a Banach space with such a representation for some even isotropic Borel measure, μ. (See, e.g. [**33**] for details.)

Volume inequalities for the body $Z_p^* = Z_p^*(\mu)$ or its polar, $Z_p = Z_p(\mu)$, that characterize the Euclidean subspaces of L_p, are easily obtained by using well-known standard inequalities (such as the Urysohn and Hölder inequalities). Much more difficult to obtain are the reverse inequalities for Z_p^* or Z_p. These have the ℓ_p^n subspaces of L_p as extremals.

In 1991 Ball [**2**] used his normalized Brascamp-Lieb inequality to obtain the sharp reverse inequality for the volume of Z_p^*, for all $p \in [1, \infty]$. Ball's inequality shows that the unit ball of ℓ_p^n is the extremal. The solution to the uniqueness problem for Ball's reverse inequality was only recently obtained by Barthe [**4**] for discrete measures by using his newly established equality conditions for the Brascamp-Lieb inequality. Barthe proved that indeed the unit ball of ℓ_p^n is the only extremal for Ball's inequality when μ is a discrete isotropic measure.

The reverse inequalities for the volume of Z_p would prove to be more resistant. Ball [**1**] established the reverse inequality for the volume of Z_p for the case $p = 1$ and predicted that for $p > 1$ these inequalities could be obtained from a reverse Brascamp-Lieb inequality. Again, the breakthrough was achieved by Barthe [**4**]. Barthe found the reverse Brascamp-Lieb inequality anticipated by Ball and used it to establish the reverse inequalities for the volume of Z_p for all $p > 1$. Barthe also established the uniqueness of the extremals when μ is a discrete measure.

Within the L_p Brunn-Minkowski theory a basic problem is how to obtain Ball's inequalities, along with their equality conditions, for isotropic measures which are not necessarily discrete. In this paper we shall derive the reverse inequalities for the volumes of both Z_p and Z_p^* and prove that the ℓ_p^n-balls are the unique extremals. All our inequalities will be obtained along with their equality conditions. This will be done for all $p \in [1, \infty]$ and all even isotropic measures, μ.

We have attempted to write an article that is simultaneously self-contained and elementary. We have given very detailed proofs. Although the questions we address arise naturally within the L_p Brunn-Minkowski theory, none of the machinery of the theory is used. The reader will find none of the staples of modern convex geometry; "L_p-curvature", "Gaussian extremals", and the "Brascamp-Lieb inequality" are conspicuous in their absence. The only inequality used to establish the reverse inequalities will be the Hölder inequality.

The ideas and techniques of Ball and Barthe play a critical role throughout this paper. It would be impossible to overstate our reliance on their work.

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0. Introduction

We assume that μ is a nonnegative finite even Borel measure on the unit sphere S^{n-1}, whose support, $\operatorname{supp}\mu$, is not contained in a subsphere of S^{n-1}. For each $p \in [1, \infty)$, define the origin-symmetric convex body $Z_p = Z_p(\mu)$ in $\mathbb{R}^n$ to be the body whose support function (see Section 1 for definition), for $u \in S^{n-1}$, is given by

$$h_{Z_p}(u)^p = \int_{S^{n-1}} |u \cdot v|^p d\mu(v), \tag{0.1}$$

and, for $p = \infty$, is given by

$$h_{Z_\infty}(u) = \lim_{p\to\infty} h_{Z_p}(u) = \max_{v \in \operatorname{supp}\mu} u \cdot v. \tag{0.2}$$

The Z_1 bodies are the *zonoids* (limits of Minkowski sums of line segments) of the classical Brunn-Minkowski theory. The Z_p bodies are the L_p*-zonoids* of the L_p Brunn-Minkowski theory.

In addition to its denoting absolute value, we shall use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$, on occasion the absolute value of the determinant of an $n \times n$ matrix, and often to denote n-dimensional volume. A Borel measure μ on S^{n-1} is *isotropic* provided

$$|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 d\mu(v), \tag{0.3}$$

for all $x \in \mathbb{R}^n$. Note that an isotropic measure cannot be concentrated on a great subsphere of S^{n-1}. In light of (0.1) and (0.3), we see that

$$\mu \text{ is isotropic if and only if } Z_2(\mu) = B, \tag{0.4}$$

where B denotes the standard unit ball in $\mathbb{R}^n$. The two most important examples of even isotropic measures on S^{n-1} are (suitably normalized) spherical Lebesgue measure and the *cross measure*. The *basic* cross measure is an even isotropic discrete measure concentrated on $\pm e_1, \dots, \pm e_n$, where $e_1, \dots, e_n$ denotes the canonical basis for $\mathbb{R}^n$. A cross measure is just a rotation of a basic cross measure; i.e., it is concentrated on $O\{\pm e_1, \dots, \pm e_n\}$, where $O \in \mathrm{O}(n)$. Note that each point in the support of a cross measure is equally weighted.

For $p \in [1, \infty]$, let $p^* \in [1, \infty]$ denote the *Hölder conjugate* of p; i.e., p^* is defined by

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

For $n, p \in (0, \infty)$, let

$$\omega_n(p) = 2^n \frac{\Gamma(1 + \frac{1}{p})^n}{\Gamma(1 + \frac{n}{p})}.$$

Let $\omega_n(\infty) = 2^n$, and abbreviate $\omega_n(2)$ by ω_n and note that for positive integer n, the unit ball of $\mathbb{R}^n$ has precisely volume ω_n. For $p \in (0,\infty)$, define c_p by

$$c_p = \left(\frac{\Gamma(1+\frac{n}{2})}{\Gamma(1+\frac{1}{2})} \frac{\Gamma(\frac{1+p}{2})}{\Gamma(\frac{n+p}{2})} \right)^{n/p},$$

and define $c_\infty = \lim_{p\to\infty} c_p = 1$.

The following two theorems are our main results.

Theorem 1. *Suppose $p \in [1,\infty]$. If μ is an even isotropic measure on S^{n-1}, then*

$$\omega_n/c_p \le |Z_p^*(\mu)| \le \omega_n(p).$$

If $p \in [1,\infty)$ is not an even integer, then there is equality in the left inequality if and only if μ is suitably normalized Lebesgue measure. For $p \ne 2$, there is equality in the right inequality if and only if μ is a cross measure.

The right inequality of Theorem 1, without the equality conditions, was proved by Ball [**2**], Proposition 8. For discrete measures the equality conditions are due to Barthe [**4**], Corollary 3.

Theorem 2. *Suppose $p \in [1,\infty]$. If μ is an even isotropic measure on S^{n-1}, then*

$$\omega_n(p^*) \le |Z_p(\mu)| \le \omega_n c_p.$$

For $p \ne 2$, there is equality in the left inequality if and only if μ is a cross measure. If $p \in [1,\infty)$ is not an even integer, then there is equality in the right inequality if and only if μ is normalized Lebesgue measure.

The left inequality, without the equality conditions, for $p = 1$ is due to Ball [**1**], Lemma 4. The left inequality of Theorem 2, for $p > 1$ is due to Barthe [**4**], Proposition 11. For discrete measures the equality conditions follow from the work of Barthe [**4**].

Theorems 1 and 2 have direct applications to volume-ratio inequalities. Recall that the *inner-volume ratio* $\mathrm{vr}_i(K)$ and the *outer-volume ratio* $\mathrm{vr}_o(K)$ of a convex body K in $\mathbb{R}^n$ are defined by

$$\mathrm{vr}_i(K) = (|K|/|E_i|)^{1/n} \quad \text{and} \quad \mathrm{vr}_o(K) = (|K|/|E_o|)^{1/n},$$

where E_i is the ellipsoid of maximal volume contained in K, and E_o is the ellipsoid of minimal volume containing K.

Let $B_p^n = \{x \in \mathbb{R}^n : |e_1 \cdot x|^p + \cdots + |e_n \cdot x|^p \le 1\}$ denote the unit ball of classical n-dimensional ℓ_p^n-space. Theorems 1 and 2 immediately give the following volume ratio inequalities.

Theorem 3. *Suppose* $1 \le p \le \infty$. *If* μ *is an isotropic measure on* S^{n-1}, *then*

$$\mathrm{vr}_i(Z_p^*) \le \mathrm{vr}_i(B_p^n) \quad \textit{and} \quad \mathrm{vr}_o(Z_p) \ge \mathrm{vr}_o(B_{p^*}^n).$$

For $p \neq 2$, *there is equality in each of the inequalities if and only if* μ *is a cross measure.*

The left inequality of Theorem 3 is due to Ball [**2**]. The right inequality is due to Barthe [**4**], for $p > 1$, and to Ball [**1**], for $p = 1$. The equality conditions for $p = \infty$ and discrete μ are due to Barthe [**4**].

1. Notation and basics regarding convex bodies

For $p \in [1, \infty]$ we shall use $|\cdot|_p$ to denote the usual ℓ_p-norm on $\mathbb{R}^n$; i.e., for $x \in \mathbb{R}^n$ and $p < \infty$

$$|x|_p = \left(\sum_{j=1}^{n} |e_j \cdot x|^p \right)^{1/p} \quad \text{and} \quad |x|_\infty = \max_{1 \le j \le n} |e_j \cdot x|.$$

We will write $\ell_p^n = (\mathbb{R}^n, |\cdot|_p)$. When $p = 2$ we write $\mathbb{R}^n$ rather than ℓ_2^n, and suppress the subscript in the norm.

Important examples of origin-symmetric convex bodies are the B_p^n, the unit balls of ℓ_p^n. Note that we have $|B_p^n| = \omega_n(p)$. When $p = 2$, we shall write simply B rather than B_2^n.

For easy subsequent referencing, we list some basic facts regarding convex bodies. See Gardner [**10**], Schneider [**42**] and Thompson [**47**] for additional details. A *convex body* is a compact convex set with nonempty interior in $\mathbb{R}^n$. In this work convex bodies always contain the origin in their interiors. A convex body K is uniquely determined by its *support function* $h_K : S^{n-1} \to (0, \infty)$ which is defined for $u \in S^{n-1}$ by

$$h_K(u) = \max\{u \cdot x : x \in K\}. \tag{1.1}$$

The *radial function* $\rho_K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ of the convex body K is defined, for $x \neq 0$, by

$$\rho_K(x) = \max\{\lambda \ge 0 \ : \ \lambda x \in K\}. \tag{1.2}$$

Using the fact that the radial function is homogeneous of degree -1, and rewriting the integrals over $\mathbb{R}^n$ as integrals over $S^{n-1} \times (0, \infty)$ in polar coordinates, shows that for each $p \in (0, \infty)$, for the volume of K we have

$$|K| = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} \exp\{-1/\rho_K(x)^p\}\, dx, \tag{1.3}$$

where the integral is with respect to Lebesgue measure on $\mathbb{R}^n$.

Associated with each origin-symmetric convex body, K, is its *polar body* K^* which may be defined by

$$K^* = \{x \in \mathbb{R}^n : h_K(x) \le 1\}. \tag{1.4}$$

This definition leads to

$$\operatorname{int} K^* = \{x \in \mathbb{R}^n : h_K(x) < 1\}. \tag{1.5}$$

The radial and support functions of K^* and K are related by:

$$\rho_{K^*} = 1/h_K \quad \text{and} \quad \rho_K = 1/h_{K^*}. \tag{1.6}$$

Note that if the convex body K is the unit ball of the Banach space $\mathbb{R}^n$ with norm $\|\cdot\|_K$, then

$$\|x\|_K = 1/\rho_K(x),$$

for all $x \in \mathbb{R}^n$.

The following duality is important: For $p \in [1, \infty]$

$$(B_p^n)^* = B_{p^*}^n.$$

For $p = 2$ this expresses the fact that $B^* = B$.

2. A basic inequality

A Borel measure (always assumed to be nonnegative and finite) μ on S^{n-1} generates a positive semi-definite $n \times n$ matrix $[\mu]$ defined by

$$[\mu] = \int_{S^{n-1}} v \otimes v \, d\mu(v)$$

or equivalently by

$$x \cdot [\mu]x = \int_{S^{n-1}} |x \cdot v|^2 d\mu(v), \tag{2.1}$$

for all $x \in \mathbb{R}^n$. In (2.1) take $x = e_j$, sum over all j, and for the trace of $[\mu]$ we get:

$$\begin{aligned} \operatorname{tr}[\mu] &= \sum_{j=1}^n e_j \cdot [\mu] e_j \\ &= \sum_{j=1}^n \int_{S^{n-1}} (e_j \cdot v)^2 d\mu(v) \\ &= \int_{S^{n-1}} 1 \, d\mu(v) \\ &= \mu(S^{n-1}). \end{aligned} \tag{2.2}$$

We shall require and review some basics regarding mixed discriminants. Recall that for positive semi-definite $n \times n$ matrices $Q_1, \dots, Q_m$ and real $\lambda_1, \dots, \lambda_m \geq 0$, the determinant of the linear combination $\lambda_1 Q_1 + \cdots + \lambda_m Q_m$ is a homogeneous polynomial of degree n in the λ_i,

$$\det(\lambda_1 Q_1 + \cdots + \lambda_m Q_m) = \sum_{1 \leq i_1, \dots, i_n \leq m} \lambda_{i_1} \cdots \lambda_{i_n} D(Q_{i_1}, \dots, Q_{i_n}),$$

where the coefficient $D(Q_{i_1}, \dots, Q_{i_n})$ depends only on $Q_{i_1}, \dots, Q_{i_n}$ (and not on any of the other Q_j) and thus may be chosen to be symmetric in its arguments. The coefficient $D(Q_{i_1}, \dots, Q_{i_n})$ is defined to be symmetric in its arguments and is called the *mixed discriminant* of $Q_{i_1}, \dots, Q_{i_n}$.

The mixed discriminant $D(Q, \dots, Q, I, \dots, I)$, with $n-k$ copies of Q and k copies of the identity matrix I will be abbreviated by $D_k(Q)$. Note that the *elementary mixed discriminants* $D_0(Q), \dots, D_n(Q)$ are thus defined as the coefficients of the polynomial

$$\det(Q + \lambda I) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i D_i(Q).$$

Obviously, $D_0(Q) = \det(Q)$ while $nD_{n-1}(Q) = \operatorname{tr}(Q)$ is the trace of Q.

For $x_1, \dots, x_n \in \mathbb{R}^n$, let $[x_1, \dots, x_n]$ denote the n-dimensional volume of the parallelotope whose defining vectors are $x_1, \dots, x_n$. We require the following easily-established (see, e.g., Busemann [**6**]) fact: Suppose $y_{ij} \in \mathbb{R}^n$, for $1 \leq i \leq m$ and $1 \leq j \leq n$. If the positive semi-definite matrices $Q_1, \dots, Q_n$ are defined by

$$x \cdot Q_j x = \sum_{i=1}^{m} |x \cdot y_{ij}|^2,$$

for all $x \in \mathbb{R}^n$, then the mixed discriminant of $Q_1, \dots, Q_n$ is given by

$$D(Q_1, \dots, Q_n) = \frac{1}{n!} \sum_{1 \leq i_1, \dots, i_n \leq m} [y_{i_1 1}, \dots, y_{i_n n}]^2.$$

Alternatively, if $\mu_1, \dots, \mu_n$ are discrete positive measures on S^{n-1} and $\operatorname{supp} \mu_j \subseteq \{v_{1j}, \dots, v_{mj}\}$, then

$$D([\mu_1], \dots, [\mu_n]) = \frac{1}{n!} \sum_{1 \leq i_1, \dots, i_n \leq m} [v_{i_1 1}, \dots, v_{i_n n}]^2 \, \mu_1(v_{i_1 1}) \cdots \mu_n(v_{i_n n}).$$

It follows that for Borel measures $\mu_1, \dots, \mu_n$ on S^{n-1}, we have

$$D([\mu_1], \dots, [\mu_n]) = \frac{1}{n!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} [v_1, \dots, v_n]^2 d\mu_1(v_1) \cdots d\mu_n(v_n). \tag{2.3}$$

We will use ν, possibly with subscripts, to denote isotropic measures exclusively. We see from definitions (2.1) and (0.3) that the matrix generated by an isotropic measure is the identity, I; i.e., $[\nu] = I$. Thus from (2.2) we see that for an isotropic measure ν,

$$\nu(S^{n-1}) = n, \tag{2.4}$$

and since $D(I, \dots, I) = \det I = 1$, from (2.3) we see that

$$1 = \frac{1}{n!}\int_{S^{n-1}}\cdots\int_{S^{n-1}}[v_1, \dots, v_n]^2 d\nu(v_1)\cdots d\nu(v_n). \tag{2.5}$$

Choose $\mu_2 = \cdots = \mu_n = \nu$. Since the measure ν is isotropic we have $[\mu_2] = \cdots = [\mu_n] = [\nu] = I$. For $u \in S^{n-1}$, choose μ_1 to be the discrete probability such that $\operatorname{supp}\mu_1 = \{u\}$. We see from (2.2) that $\operatorname{tr}[\mu_1] = 1$. Thus for the left side of (2.3) we have

$$D([\mu_1], I, \dots, I) = D_{n-1}([\mu_1]) = \frac{1}{n}\operatorname{tr}[\mu_1] = \frac{1}{n}.$$

This and the definition of μ_1, shows that (2.3) gives

$$\frac{1}{(n-1)!}\int_{S^{n-1}}\cdots\int_{S^{n-1}}[u, v_2, \dots, v_n]^2 d\nu(v_2)\cdots d\nu(v_n) \;=\; 1, \tag{2.6}$$

for each $u \in S^{n-1}$.

Suppose $t : S^{n-1} \to (0, \infty)$ is continuous, and ν is an isotropic measure on S^{n-1}. Define the measure μ by $d\mu = t\, d\nu$ and let $\mu_1 = \cdots = \mu_n = \mu$. Thus, from definition (2.1) we have

$$[\mu] = \int_{S^{n-1}} t(v)\, v \otimes v\, d\nu(v).$$

Since $D([\mu], \dots, [\mu]) = \det[\mu]$, identity (2.3) becomes

$$\begin{aligned} \det \int_{S^{n-1}} t(v)\, v \otimes v\, d\nu(v) \\ = \frac{1}{n!}\int_{S^{n-1}}\cdots\int_{S^{n-1}} t(v_1)\cdots t(v_n)\,[v_1, \dots, v_n]^2\, d\nu(v_1)\cdots d\nu(v_n). \end{aligned} \tag{2.7}$$

For a Borel measure μ, let $|f : \mu|_p$ denote the standard L_p norm of the function f with respect to μ; i.e., for $0 < p < \infty$

$$|f : \mu|_p = \Big\{\int |f|^p\, d\mu\Big\})^{1/p},$$

and, for $p = \infty$,

$$|f : \mu|_\infty = \lim_{p\to\infty} |f : \mu|_p = \sup_v |f(v)|,$$

where the supremum is to be interpreted as an essential supremum. If in addition, μ is a probability measure, then for $p = 0$ define

$$|f:\mu|_0 = \lim_{p\to 0} |f:\mu|_p = \exp \int \log |f| \, d\mu.$$

For discrete measures, the following lemma is due to Ball. Barthe [**4**] (Proposition 9) provides a very simple proof. The equality conditions are new.

The Ball-Barthe Lemma. *If $t : S^{n-1} \to (0,\infty)$ is continuous and ν is an isotropic measure on S^{n-1}, then*

$$\det \int_{S^{n-1}} t(v)\, v \otimes v \, d\nu(v) \ \geq \ \exp\left\{ \int_{S^{n-1}} \log t(v) \, d\nu(v) \right\}, \tag{2.8}$$

with equality if and only if $t(v_1)\cdots t(v_n)$ is constant for linearly independent $v_1, \dots, v_n$ in $\operatorname{supp}(\nu)$.

Proof. We first observe that by (2.6) the quantity in brackets in the right hand side of

$$\begin{aligned}
&\int_{S^{n-1}} \cdots \int_{S^{n-1}} \log t(v_1) \, [v_1, \dots, v_n]^2 \, d\nu(v_1) \cdots d\nu(v_n) \\
&\qquad = \int_{S^{n-1}} \log t(v_1) \Big[\int_{S^{n-1}} \cdots \int_{S^{n-1}} [v_1, \dots, v_n]^2 \, d\nu(v_2) \cdots d\nu(v_n) \Big] d\nu(v_1)
\end{aligned}$$

is equal to $(n-1)!$. We see from this that, for each i

$$\begin{aligned}
&\frac{1}{(n-1)!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} \log t(v_i) \, [v_1, \dots, v_n]^2 \, d\nu(v_1) \cdots d\nu(v_n) \\
&\qquad = \int_{S^{n-1}} \log t(v) \, d\nu(v).
\end{aligned}$$

Since by (2.5) the measure of the underlying space is unity, and thus the L_1-norm of the function $(v_1, \dots, v_n) \mapsto t(v_1) \cdots t(v_n)$ dominates the L_0-norm, we have together with (2.6), that

$$\begin{aligned}
&\frac{1}{n!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} t(v_1) \cdots t(v_n) \, [v_1, \dots, v_n]^2 \, d\nu(v_1) \cdots d\nu(v_n) \\
&\geq \exp\Big[\frac{1}{n!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} \log(t(v_1) \cdots t(v_n)) \, [v_1, \dots, v_n]^2 \, d\nu(v_1) \cdots d\nu(v_n) \Big] \\
&= \exp\Big[\int_{S^{n-1}} \log t(v) \, d\nu(v) \Big].
\end{aligned}$$

This together with (2.7) gives the desired inequality. Equality in the inequality holds if and only if the function $t(v_1) \cdots t(v_n)$ is constant on the support of the probability measure $\frac{1}{n!}[v_1, \dots, v_n]^2 d\nu(v_1) \cdots d\nu(v_n)$.

This is equivalent to $t(v_1)\cdots t(v_n)$ being constant for linearly independent $v_1,\dots,v_n \in \operatorname{supp}(\nu)$. See Lemma A.1 for details. q.e.d.

3. A minor inequality

Note that for even μ, from the definition (0.2) of $Z_\infty = Z_\infty(\mu)$ we see that $h_{Z_\infty}(u) \le 1$ with $h_{Z_\infty}(u) = 1$ if and only if $u \in \operatorname{supp}\mu$. Therefore Z_∞ is the smallest convex body containing $\operatorname{supp}\mu$; i.e.,

$$Z_\infty = \operatorname{conv}(\operatorname{supp}\mu). \tag{3.1}$$

For $t \in L_1(\mu)$ define $t^\circ \in \mathbb{R}^n$ by

$$t^\circ = \int_{S^{n-1}} u t(u)\, d\mu(u). \tag{3.2}$$

Obviously, for real $\lambda > 0$, we have

$$(\lambda t)^\circ = \lambda t^\circ. \tag{3.3}$$

Lemma 3.1. *Suppose $p \in [1,\infty]$, and μ is an even Borel measure on S^{n-1}. If $t \in L_{p^*}(\mu)$, then*

$$1/\rho_{Z_p}(t^\circ) \le |\, t\!:\!\mu|_{p^*}\,. \tag{3.4}$$

Proof. For $p < \infty$, define

$$M_p = \left\{t^\circ \in \mathbb{R}^n : \ |\, t\!:\!\mu|_{p^*} \le 1\right\}, \tag{3.1.1}$$

while for $p = \infty$, define M_p as the closure of this; i.e.,

$$M_\infty = \operatorname{cl}\left\{t^\circ \in \mathbb{R}^n : \ |\, t\!:\!\mu|_1 \le 1\right\}. \tag{3.1.2}$$

It is easily shown that M_p is a convex body for all $p \in [1,\infty]$.

In light of (3.3) and the fact that the radial function is homogeneous of degree -1 we see that once the desired inequality (3.4) is established for $t = t_o$, then it must hold for $t = \lambda t_o$, for all $\lambda > 0$. Therefore it is sufficient to establish (3.4) for the special case where $|t:\mu|_{p^*} = 1$. We shall do this by showing that for all $p \in [1,\infty]$,

$$M_p \subseteq Z_p. \tag{3.1.3}$$

Suppose $u \in S^{n-1}$. First note that for $p < \infty$, from (1.1) and (3.1.1), (3.2), the Hölder inequality, definition (0.1), we see that

$$\begin{aligned}
h_{M_p}(u) &= \sup_{|t:\mu|_{p^*} \le 1} u \cdot t^\circ \\
&= \sup_{|t:\mu|_{p^*} \le 1} \int_{S^{n-1}} t(v)(u \cdot v)\, d\mu(v) \\
&\le \sup_{|t:\mu|_{p^*} \le 1} |\, t\!:\!\mu|_{p^*} \left[\int_{S^{n-1}} |u \cdot v|^p d\mu(v) \right]^{1/p} \\
&= \left[\int_{S^{n-1}} |u \cdot v|^p d\mu(v) \right]^{1/p} \\
&= h_{Z_p}(u).
\end{aligned}$$

For $p = \infty$, from (1.1) and (3.1.2), (3.2), and finally (0.2), we have

$$\begin{aligned}
h_{M_\infty}(u) &= \sup_{|t:\mu|_1 \le 1} |\, u \cdot t^\circ| \\
&= \sup_{|t:\mu|_1 \le 1} \left| \int_{S^{n-1}} (u \cdot v) t(v)\, d\mu(v) \right| \\
&\le \sup_{v \in \operatorname{supp} \mu} |u \cdot v| \\
&= h_{Z_\infty}(u),
\end{aligned}$$

which establishes (3.1.3). q.e.d.

4. Characterizations of supports of measures

We shall require the following:

Lemma 4.1. *Suppose μ is an even Borel measure on S^{n-1} which is not concentrated on a great subsphere. Suppose there exists a non-constant positive function $g : \mathbb{R} \to (0, \infty)$ such that if $\{v_1, \ldots, v_n\} \subset \operatorname{supp} \mu$ and $\{v'_1, \ldots, v'_n\} \subset \operatorname{supp} \mu$ are both linearly independent sets then*

$$g(x \cdot v_1) \cdots g(x \cdot v_n) = g(x \cdot v'_1) \cdots g(x \cdot v'_n),$$

for all $x \in \mathbb{R}^n$. Then there exists a linearly independent set of vectors $\{u_1, \ldots, u_n\}$ such that

$$\operatorname{supp} \mu = \{\pm u_1, \ldots, \pm u_n\}.$$

Proof. Since μ is not concentrated on any great subsphere of S^{n-1}, there exist linearly independent $u_1, \ldots, u_n \in \operatorname{supp} \mu$. Since μ is even, $\{\pm u_1, \ldots, \pm u_n\} \subseteq \operatorname{supp} \mu$.

We argue by contradiction and assume a vector $v \in \operatorname{supp} \mu$ exists such that $v \notin \{\pm u_1, \ldots, \pm u_n\}$. Write $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$. At least one coefficient, say λ_1, is not zero. Since $\{u_1, u_2, \ldots, u_n\} \subset \operatorname{supp}\mu$ and $\{v, u_2, \ldots, u_n\} \subset \operatorname{supp}\mu$ are both linearly independent sets,

$$g(x \cdot u_1) \cdots g(x \cdot u_n) = g(x \cdot v) g(x \cdot u_2) \cdots g(x \cdot u_n),$$

for all $x \in \mathbb{R}^n$. But $g > 0$, and hence

$$g(x \cdot u_1) = g(x \cdot v),$$

for all $x \in \mathbb{R}^n$.

Since g is not constant, there are s_1 and s_2 so that $g(s_1) \neq g(s_2)$. Since u_1 and v are not parallel, there exists an $x_o \in \mathbb{R}^n$ so that $s_1 = x_o \cdot u_1$ and $s_2 = x_o \cdot v$. Thus, $g(x_o \cdot u_1) \neq g(x_o \cdot v)$ is the desired contradiction. q.e.d.

The following slight variant of Lemma 4.1 will also be needed. The almost identical proof is included for completeness.

Lemma 4.2. *Suppose μ is an even Borel measure on S^{n-1} which is not concentrated on a great subsphere. Suppose there exists a real $c > 0$ and an even positive function $g : (-c, c) \to (0, \infty)$ that is injective on $(0, c)$ such that if $\{v_1, \ldots, v_n\} \subset \operatorname{supp}\mu$ and $\{v'_1, \ldots, v'_n\} \subset \operatorname{supp}\mu$ are both linearly independent sets then*

$$g(x \cdot v_1) \cdots g(x \cdot v_n) = g(x \cdot v'_1) \cdots g(x \cdot v'_n),$$

whenever $|x| < c$. Then there exists a linearly independent set of vectors $\{u_1, \ldots, u_n\}$ such that

$$\operatorname{supp}\mu = \{\pm u_1, \ldots, \pm u_n\}.$$

Proof. Since μ is not concentrated on any great subsphere of S^{n-1}, there exist linearly independent $u_1, \ldots, u_n \in \operatorname{supp}\mu$. Since μ is even, $\{\pm u_1, \ldots, \pm u_n\} \subseteq \operatorname{supp}\mu$.

We argue by contradiction and assume a vector $v \in \operatorname{supp}\mu$ exists such that $v \notin \{\pm u_1, \ldots, \pm u_n\}$. Write $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$. At least one coefficient, say λ_1, is not zero. Since $\{v, u_2, \ldots, u_n\} \subset \operatorname{supp}\mu$ is a linearly independent set,

$$g(x \cdot u_1) \cdots g(x \cdot u_n) = g(x \cdot v) g(x \cdot u_2) \cdots g(x \cdot u_n),$$

whenever $|x| < c$. Since $g > 0$, we have

$$g(x \cdot u_1) = g(x \cdot v),$$

whenever $|x| < c$. Since g is even and injective on $(0, c)$ it follows that $x \cdot u_1 = \pm\, x \cdot v$ whenever $|x| < c$. Hence, the desired contradiction $v = \pm u_1$. q.e.d.

We shall require the following trivial observation regarding isotropic measures:

Lemma 4.3. *Suppose ν is an isotropic Borel measure on S^{n-1}. If*

$$\{u_1, \dots, u_n\} \subset S^{n-1}$$

is a basis for $\mathbb{R}^n$ and $\operatorname{supp} \nu \subseteq \{\pm u_1, \dots, \pm u_n\}$, then $\{u_1, \dots, u_n\}$ is in fact an orthonormal basis.

Proof. Since ν is isotropic, from (0.3) we have for all $x \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} a_i |x \cdot u_i|^2 = |x|^2,$$

where $a_i = \nu(\{u_i, -u_i\})$. Note that since ν is not concentrated on a great subsphere, each $a_i > 0$. Taking $x = u_j$, gives

$$\sum_{i=1}^{n} a_i |u_j \cdot u_i|^2 = 1. \tag{4.3.1}$$

This shows that $a_j \leq 1$. But from (2.4) we know $\sum_{i=1}^{n} a_i = n$ and hence, $a_j = 1$ and from (4.3.1) we see that $|u_j \cdot u_i| = 0$ for $j \neq i$. q.e.d.

5. Volume estimates of L_p zonoids

Note that from the definitions it follows immediately that if μ is a basic cross measure, then $Z_p = (B_p^n)^* = B_{p^*}^n$.

To establish Theorems 1 and 2 we first prove:

Theorem 5.1. *Suppose $p \in [1, \infty)$ and $q \in [1, \infty]$. If μ is an even isotropic measure on S^{n-1}, then*

$$|Z_p^*(\mu)|/\omega_n(p) \leq |Z_q(\mu)|/\omega_n(q^*). \tag{5.1}$$

For $(p, q) \neq (2, 2)$, equality holds if and only if there exist orthogonal unit vectors $u_1, \dots, u_n$ such that

$$\operatorname{supp} \mu = \{\pm u_1, \dots, \pm u_n\};$$

that is, μ is a cross measure.

Proof. First, assume $q \in (1, \infty]$. Define the strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\frac{1}{\Gamma(1 + \frac{1}{p})} \int_{-\infty}^{s} e^{-|\tau|^p} d\tau = \frac{1}{\Gamma(1 + \frac{1}{q^*})} \int_{-\infty}^{\phi(s)} e^{-|\tau|^{q^*}} d\tau. \tag{5.1.1}$$

Then $\phi' > 0$ and for all $s \in \mathbb{R}$

$$|s|^p = |\phi(s)|^{q^*} - \log c_{p,q} - \log \phi'(s), \tag{5.1.2}$$

where $c_{p,q} = \Gamma(1 + 1/p)/\Gamma(1 + 1/q^*)$.

Define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \int_{S^{n-1}} u\, \phi(x \cdot u)\, d\mu(u). \tag{5.1.3}$$

Now (5.1.3) and Lemma 3.1 show that

$$1/\rho_{Z_q}(Tx)^{q^*} \le \int_{S^{n-1}} |\phi(x \cdot u)|^{q^*}\, d\mu(u), \tag{5.1.4}$$

for each $x \in \mathbb{R}^n$. We see from (5.1.3) that the differential of T is given by

$$dT(x) = \int_{S^{n-1}} u \otimes u\, \phi'(x \cdot u)\, d\mu(u), \tag{5.1.5}$$

for each $x \in \mathbb{R}^n$. Thus, for all $v \in S^{n-1}$,

$$v \cdot dT(x)v = \int_{S^{n-1}} |u \cdot v|^2\, \phi'(x \cdot u)\, d\mu(u),$$

and since $\phi' > 0$, and μ is not concentrated on a great subsphere of S^{n-1}, we conclude that the matrix $dT(x)$ is positive definite, for each $x \in \mathbb{R}^n$. Hence, a simple application of the mean value theorem shows that $T : \mathbb{R}^n \to \mathbb{R}^n$ is (globally) injective.

¿From (1.3), (1.6) and definition (0.1), (5.1.2), (2.4), (5.1.5) and the Ball-Barthe Lemma (2.8), (5.1.4), making the change of variables $y = Tx$, and (1.3) again, we have:

$$\begin{aligned}
&\Gamma(1 + \tfrac{n}{p})\, |Z_p^*| \\
&= \int_{\mathbb{R}^n} \exp\{-1/\rho_{Z_p^*}(x)^p\} dx \\
&= \int_{\mathbb{R}^n} \exp\Big\{- \int_{S^{n-1}} |x \cdot u|^p\, d\mu(u)\Big\} dx \\
&= \int_{\mathbb{R}^n} \exp\Big\{- \int_{S^{n-1}} (|\phi(x \cdot u)|^{q^*} - \log \phi'(x \cdot u) - \log c_{p,q})\, d\mu(u)\Big\} dx \\
&= c_{p,q}^n \int_{\mathbb{R}^n} \exp\Big\{- \int_{S^{n-1}} |\phi(x \cdot u)|^{q^*} d\mu(u)\Big\} \exp\Big\{\int_{S^{n-1}} \log \phi'(x \cdot u)\, d\mu(u)\Big\} dx \\
&\le c_{p,q}^n \int_{\mathbb{R}^n} \exp\Big\{- \int_{S^{n-1}} |\phi(x \cdot u)|^{q^*} d\mu(u)\Big\}\ |\, dT(x)|\ \ dx \\
&\le c_{p,q}^n \int_{\mathbb{R}^n} \exp\{-1/\rho_{Z_q}(Tx)^{q^*}\}\ |\, dT(x)|\ dx \\
&\le c_{p,q}^n \int_{\mathbb{R}^n} \exp\{-1/\rho_{Z_q}(y)^{q^*}\}\, dy \\
&= c_{p,q}^n\, \Gamma(1 + \tfrac{n}{q^*})\, |Z_q|.
\end{aligned}$$

This is the desired inequality (5.1).

The case $q = 1$ of inequality (5.1) follows from the cases $q > 1$, established above, by taking the limit $q \to 1$. However, in order to establish the equality conditions, a direct proof is needed.

For $q = 1$, define $\phi : \mathbb{R} \to (-1, 1)$ by

$$\frac{1}{\Gamma(1+\frac{1}{p})} \int_{-\infty}^{s} e^{-|\tau|^p} d\tau = \int_{-\infty}^{\phi(s)} \mathbf{1}_{[-1,1]}(\tau) d\tau.$$

Observe that ϕ is a strictly increasing function and thus $\phi' > 0$. Note also that $|\phi| < 1$. For all $s \in \mathbb{R}$,

$$|s|^p = -\log c_{p,1} - \log \phi'(s), \tag{5.1.6}$$

where $c_{p,1} = \Gamma(1 + 1/p)$.

Define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \int_{S^{n-1}} u\, \phi(x \cdot u)\, d\mu(u), \tag{5.1.7}$$

for each $x \in \mathbb{R}^n$. In fact, $T : \mathbb{R}^n \to Z_1$; i.e.,

$$T(\mathbb{R}^n) \subseteq Z_1. \tag{5.1.8}$$

To see this, note that since $|\phi| \le 1$, Lemma 3.1 and (5.1.7) show that $\rho_{Z_1}(Tx) \ge 1$, for all $x \in \mathbb{R}^n$. But this and definition (1.2) immediately give us the desired $Tx \in Z_1$, for all $x \in \mathbb{R}^n$.

We see from (5.1.7) that the differential of T is given by

$$dT(x) = \int_{S^{n-1}} u \otimes u\, \phi'(x \cdot u)\, d\mu(u). \tag{5.1.9}$$

Since $\phi' > 0$, the matrix $dT(x)$ is positive definite for each $x \in \mathbb{R}^n$, and hence, a simple application of the mean value theorem shows that the transformation $T : \mathbb{R}^n \to Z_1 \subset \mathbb{R}^n$ is (globally) injective.

¿From (1.3), (0.1) and (1.6), (5.1.6), (2.4), (5.1.9) and the Ball-Barthe Lemma (2.8), and the change of variables $y = Tx$ together with (5.1.8), it follows that

$$\begin{aligned}
\Gamma(1+\tfrac{n}{p})|Z_p^*| &= \int_{\mathbb{R}^n} \exp\{-1/\rho_{Z_p^*}(x)^p\}dx \\
&= \int_{\mathbb{R}^n} \exp\Big\{-\int_{S^{n-1}} |x\cdot u|^p \, d\mu(u)\Big\}dx \\
&= \int_{\mathbb{R}^n} \exp\Big\{\int_{S^{n-1}} (\log\phi'(x\cdot u) + \log c_{p,1})\, d\mu(u)\Big\}dx \\
&= c_{p,1}^n \int_{\mathbb{R}^n} \exp\Big\{\int_{S^{n-1}} \log\phi'(x\cdot u)\, d\mu(u)\Big\}dx \\
&\le c_{p,1}^n \int_{\mathbb{R}^n} |\, dT(x)|\; dx \\
&\le c_{p,1}^n \int_{Z_1} dy \\
&= c_{p,1}^n\, |Z_1|.
\end{aligned}$$

Therefore, (5.1) holds when $q = 1$.

Assume equality holds in (5.1). The equality conditions, of the Ball-Barthe Lemma (2.8) show that this implies that for each fixed $x \in \mathbb{R}^n$, the function $\phi'(x\cdot u_1)\cdots\phi'(x\cdot u_n)$ is constant for linearly independent $u_1,\ldots,u_n \in \operatorname{supp}\mu$. If $p \neq q^*$, then the function ϕ' is not constant. Thus Lemmas 4.1 and 4.3 yield the necessity of the equality conditions for $p \neq q^*$. Setting $(p,q) = (p,2)$ and $(p,q) = (2,q)$ in (5.1) gives

$$|Z_p^*|/\omega_n(p) \le 1 \le |Z_q|/\omega_n(q^*),$$

where the first equality holds if and only if $p = 2$ or μ is a cross measure, and the second equality holds if and only if $q = 2$ or μ is a cross measure. Therefore, the equality of (5.1) holds if and only if $(p,q) = (2,2)$ or μ is a cross measure. q.e.d.

The proof above used techniques developed by Barthe in [**4**].

The next theorem follows from Theorem 5.1. The inequality in the theorem is due to Ball [**1**] for $p = 1$. For $p > 1$ the inequality is due to Barthe [**4**]. The equality conditions, for discrete measures, are due to Barthe [**4**].

Theorem 5.2. *Suppose $p \in [1,\infty]$. If μ is an even isotropic measure on S^{n-1}, then*

$$\omega_n(p^*) \le |Z_p(\mu)|, \tag{5.2}$$

and if $p \neq 2$, there is equality if and only if μ is a cross measure.

The case $p = \infty$ of Theorem 5.1 is proved in the next theorem. The inequality of the next theorem is due to Ball [**2**]. The equality conditions for discrete measures are due to Barthe [**4**].

Theorem 5.3. *Suppose $p \in [1, \infty]$. If μ is an even isotropic measure on S^{n-1}, then*

$$\omega_n(p) \geq |Z_p^*(\mu)|, \tag{5.3}$$

and for $p \neq 2$, there is equality if and only if μ is a cross measure.

Proof. The case where $p < \infty$ is a direct consequence of Theorem 5.1 with $q = 2$.

To establish the case $p = \infty$, define the strictly increasing function $\phi : (-1, 1) \to \mathbb{R}$ by

$$\int_{-1}^{t} \mathbf{1}_{[-1,1]}(\tau) d\tau = \frac{1}{\Gamma(\frac{3}{2})} \int_{-\infty}^{\phi(t)} e^{-\tau^2} d\tau. \tag{5.3.1}$$

Then $\phi' > 0$, and

$$\Gamma(\tfrac{3}{2}) \mathbf{1}_{[-1,1]}(s) = e^{-\phi(s)^2} \phi'(s), \tag{5.3.2}$$

for all $s \in (-1, 1)$. Note that ϕ' is an even function that is strictly decreasing on the interval $(-1, 0)$ and strictly increasing on the interval $(0, 1)$.

Now (0.2) and (1.5) give:

$$\operatorname{int} Z_\infty^* = \Big\{ x \in \mathbb{R}^n : \sup_{u \in \operatorname{supp} \mu} |x \cdot u| < 1 \Big\}. \tag{5.3.3}$$

We see from (5.3.3) that for each $x \in \operatorname{int} Z_\infty^*$

$$\exp\Big\{ \int_{\operatorname{supp} \mu} \log \mathbf{1}_{[-1,1]}(x \cdot u) \, d\mu(u) \Big\} = 1. \tag{5.3.4}$$

Define $T : \operatorname{int} Z_\infty^* \to \mathbb{R}^n$ by

$$Tx = \int_{S^{n-1}} u \, \phi(x \cdot u) \, d\mu(u). \tag{5.3.5}$$

Note that (5.3.3) shows that $x \cdot u$ is in the domain of ϕ, for all $x \in \operatorname{int} Z_\infty^*$ and all $u \in \operatorname{supp} \mu$. It follows from (5.3.5) and Lemma 3.1 that

$$1/\rho_{Z_2}(Tx)^2 \ \leq \int_{S^{n-1}} \phi(x \cdot u)^2 \, d\mu(u). \tag{5.3.6}$$

Now (5.3.5) gives

$$dT(x) = \int_{S^{n-1}} u \otimes u \, \phi'(x \cdot u) \, d\mu(u). \tag{5.3.7}$$

Since $\phi' > 0$, the matrix $dT(x)$ is positive definite for each $x \in \operatorname{int} Z_\infty^*$. Hence, the transformation $T : \operatorname{int} Z_\infty^* \to \mathbb{R}^n$ is (globally) injective.

Lastly, note that from (2.4), we have

$$\int_{\operatorname{supp}\mu} d\mu(u) \;=\; n. \tag{5.3.8}$$

¿From (5.3.4), (5.3.2), (5.3.8), (5.3.7) and the Ball-Barthe Lemma (2.8), (5.3.6), making the change of variable $y = Tx$, and finally (1.3), we have

$$\begin{aligned}
&\Gamma(\tfrac{3}{2})^n |Z_\infty^*| \\
&\quad = \Gamma(\tfrac{3}{2})^n \int_{\operatorname{int}(Z_\infty^*)} \exp\Big\{ \int_{\operatorname{supp}\mu} \log \mathbf{1}_{[-1,1]}(x\cdot u)\, d\mu(u) \Big\}\, dx \\
&\quad = \Gamma(\tfrac{3}{2})^n \int_{\operatorname{int}(Z_\infty^*)} \exp\Big\{ \int_{\operatorname{supp}\mu} \log[\Gamma(\tfrac{3}{2})^{-1} e^{-\phi(x\cdot u)^2} \phi'(x\cdot u)]\, d\mu(u) \Big\}\, dx \\
&\quad = \int_{\operatorname{int}(Z_\infty^*)} \exp\Big\{ \int_{\operatorname{supp}\mu} -\phi(x\cdot u)^2 d\mu(u) \Big\} \exp\Big\{ \int_{\operatorname{supp}\mu} \log \phi'(x\cdot u)\, d\mu(u) \Big\}\, dx \\
&\quad \le \int_{\operatorname{int}(Z_\infty^*)} \exp\Big\{ \int_{\operatorname{supp}\mu} -\phi(x\cdot u)^2 d\mu(u) \Big\}\, |\, dT(x)|\; dx \\
&\quad \le \int_{\operatorname{int}(Z_\infty^*)} \exp\{-1/\rho_{Z_2}(Tx)^2\}\, |\, dT(x)|\; dx \\
&\quad \le \int_{\mathbb{R}^n} e^{-1/\rho_{Z_2}(y)^2} dy \\
&\quad = \Gamma(1+\tfrac{n}{2})|B| \\
&\quad = 2^n \Gamma(\tfrac{3}{2})^n.
\end{aligned}$$

Assume equality holds in inequality (5.3) The equality conditions in the Ball-Barthe Lemma (2.8), show that this implies that there exists a $\delta > 0$ such that whenever x satisfies $|x| < \delta$, then for fixed x the function $\phi'(x\cdot u_1)\cdots\phi'(x\cdot u_n)$ is constant for linearly independent $u_1, \ldots, u_n \in \operatorname{supp}\mu$. (Note that since $Z_\infty \subseteq B$, it follows that $B \subseteq Z_\infty^*$, and hence δ may be taken to be 1.) Lemmas 4.2 and 4.3 will now yield the equality conditions. q.e.d.

In order to prove Theorems 1 and 2 of the introduction we only need:

Lemma 5.4. *Suppose $p \in [1,\infty]$. If μ is an even isotropic measure on S^{n-1}, then*

$$\omega_n/c_p \;\le\; |Z_p^*| \quad \textit{and} \quad |Z_p| \;\le\; \omega_n c_p.$$

If $p \in [1,\infty)$ is not an even integer, then there is equality in either inequality if and only if μ is suitably normalized Lebesgue measure.

To establish the lemma, first recall that for p not an even integer the L_p*-cosine transform* (see e.g. Koldobsky [**18**], [**19**], Lonke [**21**], Neyman [**36**], and Rubin [**37**], [**38**]) is injective; i.e., if $p \in [1, \infty)$ is not an even integer and the measures μ and $\bar{\mu}$ are even Borel measures on S^{n-1} such that $Z_p(\mu) = Z_p(\bar{\mu})$, then $\mu = \bar{\mu}$.

To establish the first inequality, observe that for $p < \infty$, from (1.6), definition (0.1), and the polar coordinate formula, together with the Hölder inequality, definition (0.1), interchanging the order of integration, and (2.4), we have

$$\begin{aligned}
\left(\frac{|Z_p^*|}{\omega_n}\right)^{-p/n} &= \left[\frac{1}{n\omega_n}\int_{S^{n-1}} h_{Z_p}(u)^{-n}\, du\right]^{-p/n} \\
&\le \frac{1}{n\omega_n}\int_{S^{n-1}} h_{Z_p}(u)^{p}\, du \\
&= \frac{1}{n\omega_n}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|^p d\mu(v)\, du \\
&= \frac{c_p^{p/n}}{n}\int_{S^{n-1}} d\mu(v) \\
&= c_p^{p/n},
\end{aligned}$$

with equality if and only if Z_p is a ball. The injectivity of the L_p-cosine transform now yields the equality conditions for the left inequality.

To establish the second inequality recall that the classical Urysohn inequality (see, e.g., Schneider [**42**], p. 318) states that for an origin-symmetric body K that is not a ball, the normalized volume $(|K|/\omega_n)^{1/n}$ is strictly less than the average value of its support function, h_K. Now for real p, the Urysohn inequality, followed by the Hölder inequality, definition (0.1), a change of the order of integration, and finally (2.4), gives

$$\begin{aligned}
\left(\frac{|Z_p|}{\omega_n}\right)^{1/n} &\le \frac{1}{n\omega_n}\int_{S^{n-1}} h_{Z_p}(u)\, du \\
&\le \Big[\frac{1}{n\omega_n}\int_{S^{n-1}} h_{Z_p}(u)^p\, du\Big]^{1/p} \\
&= \Big[\frac{1}{n\omega_n}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|^p d\mu(v)\, du\Big]^{1/p} \\
&= \Big[\frac{c_p^{p/n}}{n}\int_{S^{n-1}} d\mu(v)\Big]^{1/p} \\
&= c_p^{1/n},
\end{aligned}$$

with equality if and only if Z_p is a ball. The injectivity of the L_p-cosine transform now yields the equality conditions for the right inequality. q.e.d.

It is easily seen that

$$\begin{aligned} B \subseteq Z_p &\subseteq n^{\frac{1}{p}-\frac{1}{2}} B, && p \in [1,2],\\ n^{\frac{1}{p}-\frac{1}{2}} B \subseteq Z_p &\subseteq B, && p \in [2,\infty]. \end{aligned}$$

To see this note that for $p \in [1,2)$ the isotropy of μ gives us

$$\begin{aligned} h_{Z_p}(u) &= \Big[\int_{S^{n-1}} |u\cdot v|^p d\mu(v)\Big]^{\frac{1}{p}}\\ &\geq \Big[\int_{S^{n-1}} |u\cdot v|^2 d\mu(v)\Big]^{\frac{1}{p}}\\ &= 1, \end{aligned}$$

while from the Hölder inequality and the isotropy of μ we get

$$\begin{aligned} h_{Z_p}(u) &= n^{\frac{1}{p}}\Big[\frac{1}{n}\int_{S^{n-1}} |u\cdot v|^p d\mu(v)\Big]^{\frac{1}{p}}\\ &\leq n^{\frac{1}{p}}\Big[\frac{1}{n}\int_{S^{n-1}} |u\cdot v|^2 d\mu(v)\Big]^{\frac{1}{2}}\\ &= n^{\frac{1}{p}-\frac{1}{2}}. \end{aligned}$$

For $p \in (2,\infty]$ the inequalities are reversed.

The above inclusions provide an ellipsoid (in fact a ball) which contains the body Z_p (and hence is contained in the body Z_p^*). These inclusions, together with the definitions of vr_i and vr_o, and Theorems 1 and 2 give the following volume ratio inequalities:

Theorem 3. *Suppose* $1 \leq p \leq \infty$. *If* μ *is an isotropic measure on* S^{n-1}*, then*

$$\begin{aligned} \mathrm{vr}_i(Z_p^*) &\leq \mathrm{vr}_i(B_p^n),\\ \mathrm{vr}_o(Z_p) &\geq \mathrm{vr}_o(B_{p^*}^n). \end{aligned}$$

For $p \neq 2$*, there is equality in each of the inequalities if and only if* μ *is a cross measure.*

The first inequality of Theorem 3 is due to Ball [**2**]. The second inequality is due to Barthe [**4**], for $p > 1$, and to Ball [**1**], for $p = 1$. The equality conditions for $p = \infty$ and discrete μ are due to Barthe [**4**].

Appendix A.

If $x_1,\dots,x_n \in \mathbb{R}^n$, then $[x_1,\dots,x_n]$ is the n-dimensional volume of the parallelotope whose defining vectors are $x_1,\dots,x_n$. If $\mu_1,\dots,\mu_n$ are Borel measures on S^{n-1}, then let $[\mu_1\times\cdots\times\mu_n]$ denote the Borel measure on the n copies $S^{n-1}\times\cdots\times S^{n-1}$ defined by

$$\begin{aligned}\int_{S^{n-1}\times\cdots\times S^{n-1}} & f(\mathbf{u})\, d[\mu_1\times\cdots\times\mu_n](\mathbf{u}) \\ &= \int_{S^{n-1}}\cdots\int_{S^{n-1}} f(u_1,\dots,u_n)[u_1,\dots,u_n]\, d\mu_1(u_1)\dots\ d\mu_n(u_n),\end{aligned}$$

for each continuous $f : S^{n-1}\times\cdots\times S^{n-1} \longrightarrow \mathbb{R}$.

Define the set Φ by:

$$\Phi = \{(u_1,\dots,u_n) \in S^{n-1}\times\cdots\times S^{n-1} : [u_1,\dots,u_n] \neq 0\}.$$

Our goal is to establish:

Lemma A.1. *Suppose $\mu_1,\dots,\mu_n$ are finite nonnegative Borel measures on S^{n-1}, and $g : S^{n-1}\times\cdots\times S^{n-1} \longrightarrow \mathbb{R}$ is continuous. Then g is constant a.e. with respect to $[\mu_1\times\cdots\times\mu_n]$ if and only if g is constant on*

$$\Phi \cap (\operatorname{supp}\mu_1\times\cdots\times\operatorname{supp}\mu_n).$$

To prove Lemma A.1 we require the following well-known result:

Lemma A.2. *If $\mu_1,\dots,\mu_n$ are finite nonnegative Borel measures on S^{n-1}, then*

$$\operatorname{supp}\mu_1\times\cdots\times\operatorname{supp}\mu_n = \operatorname{supp}(\mu_1\times\cdots\times\mu_n).$$

Proof. Let $N_i \subset S^{n-1}$ be the largest open sets so that $\mu_i(N_i) = 0$; i.e., $N_i^c = \operatorname{supp}\mu_i$. Let $\mathcal{O}$ be the largest open set for which $\mu_1\times\cdots\times\mu_n(\mathcal{O}) = 0$; i.e.,

$$\mathcal{O}^c = \operatorname{supp}(\mu_1\times\cdots\times\mu_n).$$

First, observe that

$$\begin{aligned}&\operatorname{supp}\mu_1\times\cdots\times\operatorname{supp}\mu_n \\ &= N_1^c\times\cdots\times N_n^c \\ &= ((N_1\times S^{n-1}\times\cdots\times S^{n-1})\cup\cdots\cup(S^{n-1}\times\cdots\times S^{n-1}\times N_n))^c \\ &\supseteq \operatorname{supp}(\mu_1\times\cdots\times\mu_n).\end{aligned}$$

To see that the above inclusion cannot be strict, assume that there exists a $(u_1,\dots,u_n) \in \mathcal{O}$, such that

$$(u_1,\dots,u_n) \notin (N_1\times S^{n-1}\times\cdots\times S^{n-1})\cup\cdots\cup(S^{n-1}\times\cdots\times S^{n-1}\times N_n).$$

Thus $u_i \notin N_i$, for all i. Since $(u_1, \dots, u_n) \in \mathcal{O}$ and $\mathcal{O}$ is open, there exist open sets V_i such that $(u_1, \dots, u_n) \in V_1 \times \cdots \times V_n \subseteq \mathcal{O}$. But $u_i \in V_i$ and $u_i \notin N_i$ implies that the open sets V_i are not contained in N_i, and hence $\mu_i(V_i) > 0$. Therefore,

$$\begin{aligned} 0 &= \mu_1 \times \cdots \times \mu_n(\mathcal{O}) \\ &\geq \mu_1 \times \cdots \times \mu_n(V_1 \times \cdots \times V_n) \\ &= \mu_1(V_1) \cdots \mu_n(V_n) > 0, \end{aligned}$$

a contradiction. q.e.d.

Proof of Lemma A.1. Let $\mathcal{D}^c = \operatorname{supp}[\mu_1 \times \cdots \times \mu_n]$. Hence $\mathcal{D}^c$ is an open set such that

$$\int_{\mathcal{D}} [u_1, \dots, u_n] \, d\mu_1(u_1) \cdots d\mu_n(u_n) = 0$$

and since $[u_1, \dots, u_n] \geq 0$ on $\mathcal{D}$, we have

$$\int_{\Phi \cap \mathcal{D}} [u_1, \dots, u_n] \, d\mu_1(u_1) \cdots d\mu_n(u_n) = 0.$$

Since $[u_1, \dots, u_n] > 0$ on $\Phi \cap \mathcal{D}$, we conclude that

$$\int_{\Phi \cap \mathcal{D}} d\mu_1 \cdots d\mu_n = 0.$$

This and the fact that $\Phi \cap \mathcal{D}$ is open shows that $\Phi \cap \mathcal{D}$ is contained in the complement of the support of $\mu_1 \times \cdots \times \mu_n$. Thus,

$$\operatorname{supp}(\mu_1 \times \cdots \times \mu_n) \subseteq (\Phi \cap \mathcal{D})^c,$$

and hence

$$\Phi \cap \operatorname{supp}(\mu_1 \times \cdots \times \mu_n) \subseteq \Phi \cap (\Phi \cap \mathcal{D})^c \subseteq \mathcal{D}^c = \operatorname{supp}[\mu_1 \times \cdots \times \mu_n].$$

Thus from Lemma A.1, we have

$$\Phi \cap (\operatorname{supp} \mu_1 \times \cdots \times \operatorname{supp} \mu_n) \subseteq \operatorname{supp}[\mu_1 \times \cdots \times \mu_n]. \tag{A.1.1}$$

If g is constant a.e. with respect to $[\mu_1 \times \cdots \times \mu_n]$, then since g is continuous, g is constant on the set $\operatorname{supp}[\mu_1 \times \cdots \times \mu_n]$, and from (A.1.1) we now see that g is constant on the set $\Phi \cap (\operatorname{supp} \mu_1 \times \cdots \times \operatorname{supp} \mu_n)$.

Suppose now that g is constant on $\Phi \cap (\operatorname{supp} \mu_1 \times \cdots \times \operatorname{supp} \mu_n)$. This assumption implies that g is constant on all but a set of $[\mu_1 \times \cdots \times \mu_n]$-measure 0. To see this, simply observe that

$$\begin{aligned} &(\Phi \cap (\operatorname{supp} \mu_1 \times \cdots \times \operatorname{supp} \mu_n))^c \\ &= \Phi^c \cup (N_1^c \times \cdots \times N_n^c)^c \\ &= \Phi^c \cup (N_1 \times S^{n-1} \times \cdots \times S^{n-1}) \cup \cdots \cup (S^{n-1} \times \cdots \times S^{n-1} \times N_n), \end{aligned}$$

where $N_i^c = \operatorname{supp} \mu_i$, and clearly this is a union of sets of $[\mu_1 \times \cdots \times \mu_n]$-measure 0. q.e.d.

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