Moment-entropy inequalities for a random vector

Erwin Lutwak, Deane Yang, and Gaoyong Zhang

***Abstract*— The p-th moment matrix is defined for a real random vector, generalizing the classical covariance matrix. Sharp inequalities relating the p-th moment and Renyi entropy are established, generalizing the classical inequality relating the second moment and the Shannon entropy. The extremal distributions for these inequalities are completely characterized.**

I. Introduction

In [9] the authors demonstrated how the classical information theoretic inequality for the Shannon entropy and second moment of a real random variable could be extended to inequalities for Renyi entropy and the p-th moment. The extremals of these inequalities were also completely characterized. Moment-entropy inequalities, using Renyi entropy, for discrete random variables have also been obtained by Arikan [2].

We describe how to extend the definition of the second moment matrix of a real random vector to that of the p-th moment matrix. Using this, we extend the moment-entropy inequalities and the characterization of the extremal distributions proved in [9] to higher dimensions.

Variants and generalizations of the theorems presented can be found in work of the authors [8], [10], [11] and Bastero-Romance [3].

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II. The p-th moment matrix of a random vector

A. Basic notation

Throughout this paper we denote:

$$
\begin{aligned}
\mathbb{R}^n &= \text{n-dimensional Euclidean space}\\
x \cdot y &= \text{standard Euclidean inner product of } x, y \in R^n\\
|x| &= \sqrt{x \cdot x}\\
S &= \text{positive definite symmetric n-by-n matrices}\\
|A| &= \text{determinant of } A \in S\\
|K| &= \text{Lebesgue measure of } K \subset \mathbb{R}^n.
\end{aligned}
$$

The standard Euclidean ball in $\mathbb{R}^n$ will be denoted by B, and its volume by ω_n.

Each inner product on $\mathbb{R}^n$ can be written uniquely as

$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \langle x, y \rangle_A = Ax \cdot Ay,$$

for $A \in S$. The associated norm will be denoted by $|\cdot|_A$.

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Throughout this paper, X denotes a random vector in $\mathbb{R}^n$. The probability measure on $\mathbb{R}^n$ associated with a random vector X is denoted m_X.

We will denote the standard Lebesgue density on $\mathbb{R}^n$ by dx. By the *density function f_X of a random vector X,* we mean the Radon-Nikodym derivative of probability measure m_X with respect to Lebesgue measure.

If V is a vector space and $\Phi : \mathbb{R}^n \to V$ is a continuous function, then the expected value of $\Phi(X)$ is given by

$$E[\Phi(X)] = \int_{\mathbb{R}^n} \Phi(x)\, dm_X(x).$$

We call a random vector X *nondegenerate*, if $E[|v \cdot X|] > 0$ for each nonzero $v \in \mathbb{R}^n$.

B. The p-th moment of a random vector

For $p \in (0, \infty)$, the *standard p-th moment* of a random vector X is given by

$$E[|X|^p] = \int_{\mathbb{R}^n} |x|^p\, dm_X(x). \tag{1}$$

More generally, the p-th moment with respect to the inner product $\langle \cdot, \cdot \rangle_A$ is

$$E[|X|_A^p] = \int_{\mathbb{R}^n} |x|_A^p\, dm_X(x).$$

C. The p-th moment matrix

The second moment matrix of a random vector X is defined to be

$$M_2[X] = E[X \otimes X],$$

where for $v \in \mathbb{R}^n$, $v \otimes v$ is the linear transformation given by $x \mapsto (x \cdot v)v$. Recall that $M_2[X - E[X]]$ is the covariance matrix. An important observation is that the definition of the moment matrix does not use the inner product on $\mathbb{R}^n$.

A unique characterization of the second moment matrix is the following: Let $M = M_2[X]$. The inner product $\langle \cdot, \cdot \rangle_{M^{-1/2}}$ is the unique one whose unit ball has maximal volume among all inner products $\langle \cdot, \cdot \rangle_A$ that are normalized so that the second moment satisfies $E[|AX|^2] = n$.

We extend this characterization to a definition of the p-th moment matrix $M_p[X]$ for all $p \in (0, \infty)$.

Theorem 1: If $p \in (0, \infty)$ and X is a nondegenerate random vector in $\mathbb{R}^n$ with finite p-th moment, then there exists a unique matrix $A \in S$ such that

$$E[|X|_A^p] = n$$

and

$$|A| \geq |A'|,$$

for each $A' \in S$ such that $E[|X|_{A'}^p] = n$. Moreover, the matrix A is the unique matrix in S satisfying

$$I = E[AX \otimes AX|AX|^{p-2}].$$

We define the *p-th moment matrix* of a random vector X to be $M_p[X] = A^{-p}$, where A is given by the theorem above.

The proof of the theorem is given in §IV

III. Moment-entropy inequalities

A. Entropy

The *Shannon entropy* of a random vector X is defined to be

$$h[X] = -\int_{\mathbb{R}^n} f_X \log f_X \, dx,$$

provided that the integral above exists. For $\lambda > 0$ the *λ-Renyi entropy power* of a density function is defined to be

$$N_\lambda[X] = \begin{cases} \left(\int_{\mathbb{R}^n} f_X^\lambda\right)^{\frac{1}{1-\lambda}} & \text{if } \lambda \neq 1, \\ e^{h[f]} & \text{if } \lambda = 1, \end{cases}$$

provided that the integral above exists. Observe that

$$\lim_{\lambda \to 1} N_\lambda[X] = N_1[X].$$

The *λ–Renyi entropy* of a random vector X is defined to be

$$h_\lambda[X] = \log N_\lambda[X].$$

The entropy $h_\lambda[X]$ is continuous in λ and, by the Hölder inequality, decreasing in λ. It is strictly decreasing, unless X is a uniform random vector.

It follows by the chain rule that

$$N_\lambda[AX] = |A| N_\lambda[X], \tag{2}$$

for each $A \in S$.

B. Relative entropy

Given two random vectors X, Y in $\mathbb{R}^n$, their *relative Shannon entropy* or *Kullback–Leibler distance* [6], [5], [1] (also, see page 231 in [4]) is defined by

$$h_1[X, Y] = \int_{\mathbb{R}^n} f_X \log\left(\frac{f_X}{f_Y}\right) dx, \tag{3}$$

provided that the integral above exists. Given $\lambda > 0$, we define the *relative λ–Renyi entropy power of X and Y* as follows. If $\lambda \neq 1$, then

$$N_\lambda[X, Y] = \frac{\left(\int_{\mathbb{R}^n} f_Y^{\lambda-1} f_X \, dx\right)^{\frac{1}{1-\lambda}} \left(\int_{\mathbb{R}^n} f_Y^\lambda \, dx\right)^{\frac{1}{\lambda}}}{\left(\int_{\mathbb{R}^n} f_X^\lambda \, dx\right)^{\frac{1}{\lambda(1-\lambda)}}}, \tag{4}$$

and, if $\lambda = 1$, then

$$N_1[X, Y] = e^{h_1[X,Y]},$$

provided in both cases that the righthand side exists. Define the *λ–Renyi relative entropy of random vectors X and Y* by

$$h_\lambda[X, Y] = \log N_\lambda[X, Y].$$

Observe that $h_\lambda[X, Y]$ is continuous in λ.

Lemma 2: If X and Y are random vectors such that $h_\lambda[X]$, $h_\lambda[Y]$, and $h_\lambda[X, Y]$ are finite, then

$$h_\lambda[X, Y] \geq 0.$$

Equality holds if and only if $X = Y$.

Proof: If $\lambda > 1$, then by the Hölder inequality,

$$\int_{\mathbb{R}^n} f_Y^{\lambda-1} f_X \, dx \leq \left(\int_{\mathbb{R}^n} f_Y^\lambda \, dx\right)^{\frac{\lambda-1}{\lambda}} \left(\int_{\mathbb{R}^n} f_X^\lambda \, dx\right)^{\frac{1}{\lambda}},$$

and if $\lambda < 1$, then we have

$$\begin{aligned} \int_{\mathbb{R}^n} f_X^\lambda &= \int_{\mathbb{R}^n} (f_Y^{\lambda-1} f_X)^\lambda f_Y^{\lambda(1-\lambda)} \\ &\leq \left(\int_{\mathbb{R}^n} f_Y^{\lambda-1} f_X\right)^\lambda \left(\int_{\mathbb{R}^n} f_Y^\lambda\right)^{1-\lambda}. \end{aligned}$$

The inequality for $\lambda = 1$ follows by taking the limit $\lambda \to 1$.

The equality conditions for $\lambda \neq 1$ follow from the equality conditions of the Hölder inequality. The inequality for $\lambda = 1$, including the equality condition, follows from the Jensen inequality (details may be found, for example, page 234 in [4]). ∎

C. Generalized Gaussians

We call the extremal random vectors for the moment-entropy inequalities *generalized Gaussians* and recall their definition here.

Given $t \in \mathbb{R}$, let

$$t_+ = \max(t, 0).$$

Let

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx$$

denote the Gamma function, and let

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

denote the Beta function.

For each $p \in (0, \infty)$ and $\lambda \in (n/(n+p), \infty)$, define the *standard generalized Gaussian* to be the random vector Z in $\mathbb{R}^n$ whose density function $f_Z : \mathbb{R}^n \to [0, \infty)$ is given by

$$f_Z(x) = \begin{cases} a_{p,\lambda}(1 + (1-\lambda)|x|^p)_+^{1/(\lambda-1)} & \text{if } \lambda \neq 1 \\ a_{p,1} e^{-|x|^p} & \text{if } \lambda = 1, \end{cases} \tag{5}$$

where

$$a_{p,\lambda} = \begin{cases} \dfrac{p(1-\lambda)^{\frac{n}{p}}}{n\omega_n \beta(\frac{n}{p}, \frac{1}{1-\lambda} - \frac{n}{p})} & \text{if } \lambda < 1, \\ \dfrac{p}{n\omega_n \Gamma(\frac{n}{p})} & \text{if } \lambda = 1, \\ \dfrac{p(\lambda-1)^{\frac{n}{p}}}{n\omega_n \beta(\frac{n}{p}, \frac{\lambda}{\lambda-1})} & \text{if } \lambda > 1. \end{cases}$$

Any random vector Y in $\mathbb{R}^n$ that can be written as $Y = AZ$, for some $A \in S$ is called a *generalized Gaussian*.

D. Information measures of generalized Gaussians

If $0 < p < \infty$ and $\lambda > n/(n+p)$, the λ-Renyi entropy power of the standard generalized Gaussian random vector Z is given by

$$N_\lambda[Z] = \begin{cases} \left(1 + \frac{n(\lambda-1)}{p\lambda}\right)^{\frac{1}{\lambda-1}} a_{p,\lambda}^{-1} & \text{if } \lambda \neq 1 \\ e^{\frac{n}{p}} a_{p,1}^{-1} & \text{if } \lambda = 1 \end{cases}$$

If $0 < p < \infty$ and $\lambda > n/(n+p)$, then the p-th moment of Z is given by

$$E[|Z|^p] = \left[\lambda\left(1 + \frac{p}{n}\right) - 1\right]^{-1}.$$

We define the constant

$$\begin{aligned} c(n,p,\lambda) &= \frac{E[|Z|^p]^{1/p}}{N_\lambda[Z]^{1/n}} \\ &= a_{p,\lambda}^{1/n}\left[\lambda\left(1+\frac{p}{n}\right) - 1\right]^{-\frac{1}{p}} b(n,p,\lambda), \end{aligned} \tag{6}$$

where

$$b(n,p,\lambda) = \begin{cases} \left(1 - \frac{n(1-\lambda)}{p\lambda}\right)^{\frac{1}{n(1-\lambda)}} & \text{if } \lambda \neq 1 \\ e^{-1/p} & \text{if } \lambda = 1. \end{cases}$$

Observe that if $\lambda \neq 1$ and $0 < p < \infty$, then

$$\int_{\mathbb{R}^n} f_Z^\lambda = a_{p,\lambda}^{\lambda-1}(1 + (1-\lambda)E[|Z|^p]), \tag{7}$$

and if $\lambda = 1$, then

$$h[Z] = -\log a_{p,1} + E[|Z|^p]. \tag{8}$$

We will also need the following scaling identities:

$$f_{tZ}(x) = t^{-n} f_Z(t^{-1}x), \tag{9}$$

for each $x \in \mathbb{R}^n$. Therefore,

$$\int_{\mathbb{R}^n} f_{tZ}^\lambda \, dx = t^{n(1-\lambda)} \int_{\mathbb{R}^n} f_Z^\lambda \, dx, \tag{10}$$

and

$$E[|tZ|^p] = t^p E[|Z|^p].$$

E. Spherical moment-entropy inequalities

The proof of Theorem 2 in [9] extends easily to prove the following. A more general version can be found in [7].

Theorem 3: If $p \in (0,\infty)$, $\lambda > n/(n+p)$, and X is a random vector in $\mathbb{R}^n$ such that $N_\lambda[X], E[|X|^p] < \infty$, then

$$\frac{E[|X|^p]^{1/p}}{N_\lambda[X]^{1/n}} \geq c(n,p,\lambda),$$

where $c(n,p,\lambda)$ is given by (6). Equality holds if and only if $X = tZ$, for some $t \in (0,\infty)$.

Proof: For convenience let $a = a_{p,\lambda}$. Let

$$t = \left(\frac{E[|X|^p]}{E[|Z|^p]}\right)^{1/p} \tag{11}$$

and $Y = tZ$.

If $\lambda \neq 1$, then by (9) and (5), (1), (11), and (7),

$$\begin{aligned} &\int_{\mathbb{R}^n} f_Y^{\lambda-1} f_X \\ &\geq a^{\lambda-1}t^{n(1-\lambda)} + (1-\lambda)a^{\lambda-1}t^{n(1-\lambda)-p}\int_{\mathbb{R}^n} |x|^p f_X(x)\,dx \\ &= a^{\lambda-1}t^{n(1-\lambda)}(1 + (1-\lambda)t^{-p}E[|X|^p]) \\ &= a^{\lambda-1}t^{n(1-\lambda)}(1 + (1-\lambda)E[|Z|^p]) \\ &= t^{n(1-\lambda)}\int_{\mathbb{R}^n} f_Z^\lambda, \end{aligned} \tag{12}$$

where equality holds if $\lambda < 1$. It follows that if $\lambda \neq 1$, then by Lemma 2, (4), (10) and (12), and (11), we have

$$\begin{aligned} 1 &\leq N_\lambda[X,Y]^\lambda \\ &= \left(\int_{\mathbb{R}^n} f_Y^\lambda\right)\left(\int_{\mathbb{R}^n} f_X^\lambda\right)^{-\frac{1}{1-\lambda}}\left(\int_{\mathbb{R}^n} f_Y^{\lambda-1}f_X\right)^{\frac{\lambda}{1-\lambda}} \\ &\leq t^n \frac{N_\lambda[Z]}{N_\lambda[X]} \\ &= \frac{E[|X|^p]^{n/p}}{N_\lambda[X]}\frac{N_\lambda[Z]}{E[|Z|^p]^{n/p}}. \end{aligned}$$

If $\lambda = 1$, then by Lemma 2, (3) and (5), and (8) and (11),

$$\begin{aligned} 0 &\leq h_1[X,Y] \\ &= -h[X] - \log a + n\log t + t^{-p}E[|X|^p] \\ &= -h[X] + h[Z] + \frac{n}{p}\log\frac{E[|X|^p]}{E[|Z|^p]}. \end{aligned}$$

Lemma 2 shows that equality holds in all cases if and only if $Y = X$. ∎

F. Elliptic moment-entropy inequalities

Corollary 4: If $A \in S$, $p \in (0,\infty)$, $\lambda > n/(n+p)$, and X is a random vector in $\mathbb{R}^n$ satisfying $N_\lambda[X], E[|X|^p] < \infty$, then

$$\frac{E[|X|_A^p]^{1/p}}{|A|^{1/n}N_\lambda[X]^{1/n}} \geq c(n,p,\lambda), \tag{13}$$

where $c(n,p,\lambda)$ is given by (6). Equality holds if and only if $X = tA^{-1}Z$ for some $t \in (0,\infty)$.

Proof: By (2) and Theorem 3,

$$\begin{aligned} \frac{E[|X|_A^p]^{1/p}}{|A|^{1/n}N_\lambda[X]^{1/n}} &= \frac{E[|AX|^p]^{1/p}}{N_\lambda[AX]^{1/n}} \\ &\geq \frac{E[|Z|^p]^{1/p}}{N_\lambda[Z]^{1/n}}, \end{aligned}$$

and equality holds if and only if $AX = tZ$ for some $t \in (0,\infty)$. ∎

G. Affine moment-entropy inequalities

Optimizing Corollary 4 over all $A \in S$ yields the following affine inequality.

Theorem 5: If $p \in (0,\infty)$, $\lambda > n/(n+p)$, and X is a random vector in $\mathbb{R}^n$ satisfying $N_\lambda[X], E[|X|^p] < \infty$, then

$$\frac{|M_p[X]|^{1/p}}{N_\lambda[X]} \geq n^{-n/p}c(n,p,\lambda)^n,$$

where $c(n,p,\lambda)$ is given by (6). Equality holds if and only if $X = A^{-1}Z$ for some $A \in S$.

Proof: Substitute $A = M_p[X]^{-1/p}$ into (13) ■

Conversely, Corollary 4 follows from Theorem 5 by Theorem 1.

IV. Proof of Theorem 1

A. Isotropic position of a probability measure

A Borel measure μ on $\mathbb{R}^n$ is said to be *in isotropic position*, if

$$\int_{\mathbb{R}^n} \frac{x \otimes x}{|x|^2}\, d\mu(x) = \frac{1}{n} I, \tag{14}$$

where I is the identity matrix.

Lemma 6: If $p \geq 0$ and μ is a Borel probability measure in isotropic position, then for each $A \in S$,

$$|A|^{-1/n} \left(\int_{\mathbb{R}^n} \frac{|Ax|^p}{|x|^p}\, d\mu(x) \right)^{1/p} \geq 1,$$

with either equality holding if and only if $A = aI$ for some $a > 0$.

Proof: By Hölder's inequality,

$$\left(\int_{\mathbb{R}^n} \frac{|Ax|^p}{|x|^p}\, d\mu(x) \right)^{1/p} \geq \exp\left(\int_{\mathbb{R}^n} \log \frac{|Ax|}{|x|}\, d\mu(x) \right),$$

so it suffices to prove the $p = 0$ case only.

By (14),

$$\int_{\mathbb{R}^n} \frac{(x \cdot e)^2}{|x|^2}\, d\mu(x) = \frac{1}{n}, \tag{15}$$

for any unit vector e.

Let $e_1, \ldots, e_n$ be an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. By the concavity of log, and (15),

$$\begin{aligned}
\int_{\mathbb{R}^n} \log \frac{|Ax|}{|x|}\, d\mu(x) &= \frac{1}{2} \int_{\mathbb{R}^n} \log \frac{|Ax|^2}{|x|^2}\, d\mu(x) \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \log \sum_{i=1}^n \lambda_i^2 \frac{(x \cdot e_i)^2}{|x|^2}\, d\mu(x) \\
&\geq \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{(x \cdot e_i)^2}{|x|^2} \log \lambda_i^2\, d\mu(x) \\
&= \log |A|^{1/n}.
\end{aligned}$$

The equality condition follows from the strict concavity of log. ■

B. Proof of theorem

Lemma 7: If $p > 0$ and X is a nondegenerate random vector in $\mathbb{R}^n$ with finite p-th moment, then there exists $c > 0$ such that

$$E[|e \cdot X|^p] \geq c, \tag{16}$$

for every unit vector e.

Proof: The left side of (16) is a positive continuous function of the unit sphere, which is compact. ■

Theorem 8: If $p \geq 0$ and X is a nondegenerate random vector in $\mathbb{R}^n$ with finite p-th moment, then there exists $A \in S$, unique up to a scalar multiple, such that

$$|A|^{-1/n} E[|AX|^p]^{1/p} \leq |A'|^{-1/n} E[|A'X|^p]^{1/p} \tag{17}$$

for every $A' \in S$.

Proof: Let $S' \subset S$ be the subset of matrices whose maximum eigenvalue is exactly 1. This is a bounded set inside the set of all symmetric matrices, with its boundary $\partial S'$ equal to positive semidefinite matrices with maximum eigenvalue 1 and minimum eigenvalue 0. Given $A' \in S'$, let e be an eigenvector of A' with eigenvalue 1. By Lemma 7,

$$\begin{aligned}
|A'|^{-1/n} E[|A'X|^p]^{1/p} &\geq |A'|^{-1/n} E[|X \cdot e|^p]^{1/p} \\
&\geq c^{1/p} |A'|^{-1/n}.
\end{aligned} \tag{18}$$

Therefore, if A' approaches the boundary $\partial S'$, the left side of (18) grows without bound. Since the left side of (18) is a continuous function on S', the existence of a minimum follows.

Let $A \in S$ be such a minimum and $Y = AX$. Then for each $B \in S$,

$$\begin{aligned}
|B|^{-1/n} E[|BY|^p]^{1/p} &= |A|^{1/n} |BA|^{-1/n} E[|(BA)X|^p]^{1/p} \\
&\geq |A|^{1/n} |A|^{-1/n} E[|AX|^p]^{1/p} \\
&= E[|Y|^p]^{1/p}.
\end{aligned} \tag{19}$$

with equality holding if and only if equality holds for (17) with $A' = BA$. Setting $B = I + tB'$ for $B' \in S$, we get

$$|I + tB'|^{-1/n} E[|(I + tB')Y|^p]^{1/p} \geq E[|Y|^p]^{1/p},$$

for each t near 0. It follows that

$$\left. \frac{d}{dt} \right|_{t=0} |I + tB'|^{-1/n} E[|(I + tB')Y|^p]^{1/p} = 0,$$

for each $B' \in S$. A straightforward computation shows that this holds only if

$$\frac{1}{n} E[|Y|^p] I = E[Y \otimes Y |Y|^{p-2}]. \tag{20}$$

Applying Lemma 6 to

$$d\mu(x) = \frac{|x|^p\, dm_Y(x)}{nE[|Y|^p]},$$

implies that equality holds for (19) only if $B = aI$ for some $a \in (0, \infty)$. This, in turn, implies that equality holds for (17) only if $A' = aA$. ■

Theorem 1 follows from Theorem 8 by rescaling A so that $E[|Y|^p] = n$ and substituting $Y = AX$ into (20).

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