ON WILHELM BLASCHKE'S MATHEMATICAL WORK AND INFLUENCE ON S. S. CHERN

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This essay is based on the obituary written by I. M. Yaglom [18] and Chern's essay [7] on Blaschke's contributions to mathematics.

1. Blaschke's life

Blaschke was born in Graz, Austria in 1885. His early mathematical training was by his father, who taught secondary school geometry. Later, he studied from many distinguished geometers, including Wirtinger, Study, Bianchi, Engel, Hilbert, and Klein. In 1919 he was appointed Professor at the newly founded university in Hamburg and remained there for the rest of his life. Among his colleagues there were von Neumann, Siegel, Artin, Ostrowski, Rademacher, Radon, Hecke, Hasse, Kollatz, Nielsen, Schreier, and Sperner. His best known students were Luis Santalo and S. S. Chern.

2. The mathematical work of Blaschke

Blaschke's best known work is in convex geometry, affine differential geometry, and integral geometry.

2.1. Convex geometry.

In convex geometry, Blaschke established a compactness theorem for sequences of convex bodies, now known as the Blaschke selection theorem, and used to prove new sharp convex geometric inequalities. It states that any sequence of convex sets contained in a bounded set has a subsequence that converges with respect to the Hausdorff metric. This result continues to be a useful tool for establishing sharp isoperimetric-type inequalities satisfied by convex bodies.

Blaschke also formulated what is now known as the Blaschke-Santalo inequality, which is a fundamental affine geometric inequality for convex bodies. It has deep connections to probability and functional analysis, as well as number theory, partial differential equations, and differential geometry. Generalizations of the Blaschke-Santalo inequality are still actively studied today. The inequality states that given a convex body $K \subset \mathbb{R}^n$ with its center of mass at the origin and its polar body K^*, their volumes satisfy the inequality $V(K)V(K^*) \leq V(B)^2$, where B is the standard Euclidean ball, and equality holds if and only if K is an ellipsoid. Blaschke established this when $n \leq 3$, and Santalo [15] extended it to all dimensions.

One of the most important outstanding unsolved problems in convex geometry is the Mahler conjecture, which states that there is a sharp reverse inequality, where equality holds if and only if the body is a simplex. This conjecture has been established only under additional assumptions and is still actively studied.

2.2. **Integral geometry.** Work in integral geometry dates back at least to the work of Crofton, who showed that the invariant measure of the set of lines intersecting an arc in the plane is proportional to the arclength. Blaschke, however, was the first to view it as a subject just as important as differential geometry. He initiated an effort to develop the foundations of the subject, which was continued by his students Chern and Santalo.

In particular, Blaschke initiated a systematic study of kinematic formulas. A kinematic formula can be described as follows: Let G_1 and G_2 be geometric objects (a linear subspace, a submanifold, or a subdomain) in $\mathbb{R}^n$. Each has natural geometric invariants associated with it, including Euler characteristic, volume, and volume of the

boundary. On the other hand, an integral geometric invariant of G_2 can be defined by averaging over all rigid motions a geometric invariant of the intersection of G_2 with a rigid motion of G_1. A kinematic formula expresses the latter as a linear combination of the former.

Blaschke's work focused on Euclidean space, but kinematic formulas can be generalized to other geometric structures defined by a transformation group. Much work was done by Chern, Santalo, and others to develop these generalizations. See, for example, the book of Santalo [16]. Chern, in papers such as [5] and [6], showed how kinematic formulas could be derived on homogeneous spaces.

2.3. **Affine differential geometry.** Blaschke also initiated the study of affine differential geometry following Klein's Erlangen Program. In the second volume [2] of his series of monographs on differential geometry, he systematically derives local differential geometric invariants for a submanifold of $\mathbb{R}^n$ that are invariant or behave nicely under affine transformations. He is best known for introducing the notion of an affine normal of a hypersurface in $\mathbb{R}^n$ for $n \geq 3$. The affine normal is an affine analogue to the Gauss map in Euclidean differential geometry. This can be used to define the notion of an affine sphere, which can also be described as a solution to a Monge-Ampère-type PDE. See Loftin [13] and Loftin-Wang-Yang [14] for a survey on affine spheres.

2.4. **Riemannian geometry.** Blaschke's main contributions to Riemannian geometry consist of his expository writings and questions he posed for future mathematicians to study. The best known example of this is the Blaschke conjecture.

Blaschke [3] introduced the notion of a wiedersehen surface. A closed 2-dimensional Riemannian manifold M is *wiedersehen*, if there exists $d > 0$ such that for each $p \in M$ there is another point $q \in M$ such that every geodesic starting at p passes through q at distance d. Blaschke [3] conjectured in 1921 that any wiedersehen surface must be the 2-sphere with a constant curvature Riemannian metric. Chern [7] describes the early history of this conjecture. It was proved by Green [12].

The definition of a wiedersehen surface extends without change to that of a wiedersehen manifold in higher dimensions. The question of whether the Blaschke conjecture holds in higher dimensions remained open until relatively recently. In Appendix D of the book [1] by Besse (a pseudonym of Marcel Berger), Berger uses an inequality of Kazdan (in Appendix E of [1]) to show that the volume of a wiedersehen n-manifold is bounded from below by the volume of the standard n-sphere with radius $r = d/2pi$. Weinstein [17] showed that the volume of the wiedersehen manifold M is given by a cohomological computation on the space of closed geodesics on M and used this to establish the Blaschke conjecture in even dimensions. C. T. Yang [19] carried out the cohomological computation in odd dimensions, completing the proof of the Blaschke conjecture.

3. Blaschke's influence on Chern

3.1. **Chern's early years.** S. S. Chern first met Blaschke in 1932 when Blaschke visited Peking where Chern was a young college student. According to Chern [7], Blaschke's "insistence mathematics to be a lively and intelligible subject" was instrumental in Chern's decision to study mathematics in Hamburg. Chern went to Hamburg in 1934 and received his doctorate under the supervision of Blaschke in 1936. Chern also began to study exterior differential systems and what is now known as Cartan-Kähler theory with Kähler. Blaschke then arranged for Chern to spend a year in Paris with Elie Cartan to continue his studies.

Chern was able to use exterior differential forms quite effectively to extend Blaschke's ideas in differential and integral geometry to a more abstract framework. Calculations like this led to Chern's work on the volumes of tubes and eventually characteristic classes.

Inspired by earlier work of Lie and Poincar; Blaschke and his student Bol [4] studied web geometry. Chern and Griffiths [10, 11, 9] did some work on the subject. See, for example, the survey of Chern [8] for more details.

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