GROMOV'S APPROACH TO SOLVING UNDERDETERMINED SYSTEMS OF PDE'S

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In §2 of Gromov's book *Partial Differential Relations* [1], he proves that there exist many linear inhomogeneous underdetermined systems of PDE's of the form

$$Du = f$$

that can be solved with an arbitrary righthand side f by *differentiating* f rather than using an integral operator, which is the usual practice. This is completely counterintuitive for people working in analysis.

In §1 I describe an example of how this works. In subsequent sections I describe how Gromov's approach can be explained and extended using Pfaff's theorem for a nondegenerate differential 1-form. I would like to thank Robert Bryant for explaining all of this to me.

1. Examples

A trivial example is the underdetermined PDE

$$\sum_{i=1}^{n} a_i \partial_i u + v = f,$$

which can be solved by simply setting $u = 0$ and $v = f$. Here, we work through a more interesting example that Gromov often mentions.

Given smooth functions a and b satisfying a non-degeneracy condition (to be determined below), we want, for any smooth function f on $\mathbb{R}$, to find explicit pointwise formulas (in terms of a, b, and f) for functions u and v on a neighborhood of $0 \in \mathbb{R}$ that satisfy the ODE

$$au' + bv' = f, \tag{1}$$

The differential operator on the left is

$$D = \begin{bmatrix} a & b \end{bmatrix} \frac{d}{dx},$$

and the ODE can be written as

$$D \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \end{bmatrix}.$$

To solve the PDE for any function f, it suffices to find a right inverse $R : C^\infty((-\delta, \delta)) \to C^\infty((-\delta, \delta), \mathbb{R}^2)$, for some $\delta > 0$, such that

$$DR = I. \tag{2}$$

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Gromov's brilliant insight is that many underdetermined partial differential operators have a right inverse that is also a partial differential operator and that the right inverse can be found by a pointwise algebraic calculation.

For this particular example, we can solve for R assuming that it is a zero-th differential operator, i.e. a matrix-valued function

$$R = \begin{bmatrix} p \\ q \end{bmatrix}.$$

However, if we use (2) directly, we get a system of ODE's for the components of R, which has only made the problem more difficult and not easier.

The trick is to note that (2) holds if and only if the formal adjoints of R and D satisfy

$$R^* D^* = I, \tag{3}$$

where

$$D^* f = \frac{d}{dx} \begin{bmatrix} a \\ b \end{bmatrix} f = \begin{bmatrix} af' + a'f \\ bf' + b'f \end{bmatrix}$$

and

$$R^* = \begin{bmatrix} p & q \end{bmatrix}.$$

We therefore want to solve for p and q such that for any f,

$$\begin{aligned} f &= R^* D^* f \\ &= p(af' + a'f) + q(bf' + b'f) \\ &= (pa + qb)f' + (pa' + qb')f. \end{aligned}$$

This in turn is equivalent to the equations

$$\begin{aligned} pa + qb &= 0 \\ pa' + qb' &= 1, \end{aligned}$$

which, if

$$0 \neq \det \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} = ab' - a'b, \tag{4}$$

is easily solved:

$$p = \frac{-b}{ab' - a'b}, \quad q = \frac{a}{ab' - a'b}.$$

Unwinding everything, this implies that for any function f the functions

$$u = \frac{-b}{ab' - a'b} f \text{ and } v = \frac{a}{ab' - a'b} f \tag{5}$$

satisfy (1). Note that verifying this directly requires a calculation whose outcome is not obvious.

The nondegeneracy condition required here is that

$$a'b - ab' \neq 0.$$

This is the Wronskian of a and b, and a natural question is why this is relevant. Recall that if a is nonzero on an interval, then the Wronskian vanishes if and only if $b = ca$ for some constant c. Therefore, the ODE is of the form

$$(u + cv)' = \frac{f}{a},$$

which can be solved only via integration.

2. Generalization to underdetermined systems of PDE's

In general, a differential operator R^* of some order satisfying (3) exists if and only if the coefficients of R^* satisfy a pointwise linear system of equations, whose coefficients are the coefficients of D^* and their derivatives. Therefore, if the number of equations in this system is less than or equal to the number of unknowns (the number of coefficients of R) and the system has maximal rank, then a solution always exists.

Gromov shows that if the number of equations in the original system is sufficently less than the number of unknown functions, then the system (3) for a generic differential operator has maximal rank and therefore (3) has a solution.

He uses this result in Chapter 3 of [1] to reduce the codimension needed for the existence of local isometric embeddings of a generic Riemannian metic and global isometric embeddings of generic Riemannian metrics on manifolds that have certain specified topological types such as the torus.

3. Solving an underdetermined ODE via Pfaff

Robert Bryant was kind enough to explain how the approach described in §1 can be formulated and extended using exterior differential systems. The overall point is that solutions to (1) correspond to integral curves of a 1-form, which is a contact form if (4) holds. Pfaff's theorem states that there exist co-ordinates for which the contact form is in normal form. Here, these co-ordinates can be written down explicitly. These formulas can in turn be used to obtain *all* solutions to (1) using only formal differentiation and linear algebra.

In fact, everything presented here, both Gromov's approach and the one described below, works for any first order linear ODE with two unknown functions:

$$a_1 u' + a_0 u + b_1 v' + b_0 v = f. \tag{6}$$

I show below how to find all solutions to this ODE.

Given co-ordinates x, u, v on $\mathbb{R}^3$, set

$$\begin{aligned} \theta &= a_1\, du + b_1\, dv - (f - a_0 u - b_0 v)\, dx \qquad (7)\\ &= d(au + bv) - (f + (a_1' - a_0)u + (b_1' - b_0)v)\, dx \\ &= dy - p\, dx, \qquad (8) \end{aligned}$$

where

$$y = au + bv \text{ and } p = f + Au + Bv \tag{9}$$

and

$$A = a_1' - a_0 \text{ and } B = b_1' - b_0.$$

A straightforward calculation shows that

$$dx \wedge dy \wedge dp = -\theta \wedge d\theta = (a_1 B - b_1 A) dx \wedge du \wedge dv.$$

Therefore, x, y, p are co-ordinates on $\mathbb{R}^3$ if and only if $\theta \wedge d\theta$ is nonzero everywhere, which in turn holds if and only

$$a_1 B - b_1 A \neq 0, \tag{10}$$

The functions $u(x)$ and $v(x)$ give a solution to (1) if and only if the curve

$$C = \{(x, u(x), v(x)) \ : \ x \in \mathbb{R}\} \subset \mathbb{R}^3$$

is an integral curve of θ on which dx is nowhere vanishing. On the other hand, x, y, p are also co-ordinates on this space if and only if

$$\begin{aligned} 0 &\neq dx \wedge dy \wedge dp \\ &= dx \wedge (a_1' u\, dx + a_1\, du + b_1' v\, dx + b_1\, dv) \wedge ((f' + A'u + B'v)\, dx + A\, du + B\, dv) \\ &= (a_1 B - b_1 A) dx \wedge du \wedge dv. \end{aligned}$$

If we assume the nondegeneracy condition then θ vanishes on a curve C if and only if there exists a function $g(x)$ such that for any $x \in \mathbb{R}$,

$$(x, y, p) = (x, g(x), g'(x)) \in C,$$

Therefore, functions u and v solve (6) if and only if

$$\begin{bmatrix} a_1 & b_1 \\ A & B \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} g \\ g' - f \end{bmatrix},$$

which holds if and only if

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{a_1 B - b_1 A} \begin{bmatrix} B & -b_1 \\ -A & a_1 \end{bmatrix} \begin{bmatrix} g \\ g' - f \end{bmatrix}.$$

The ODE (1) is the special case where $a_0 = b_0 0$, and Gromov's solution (5) is the one given by setting $g = 0$.

4. Underdetermined first order quasilinear ODE

All of this can be generalized to a first order quasilinear ODE for two functions,

$$a(x, u, v)u' + b(x, u, v)v' = f(x, uv),$$

but using the implicit function theorem instead of explicit formulas to solve for u and v in terms of an arbitrary function g.

Pfaff's theorem states that an 1-form θ on $\mathbb{R}^3$ can be written in local-ordinates x, y, p as

$$\theta = dy - p\, dx,$$

if and only if $\theta \wedge d\theta \neq 0$. Since

$$\theta \wedge \, d\theta = -dx \wedge dy \wedge dp,$$

this holds holds if and only if $b_1 A - a_1 B \neq 0$. Here, Pfaff's theorem is not needed, because (9) provide explicit formulas for y and p satisfying (8).

References

[1] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9, Springer-Verlag, Berlin, 1986.