

## Abel's View of Abel's Theorem

Consciously or unconsciously, we historians of mathematics tend to think that we now know how to do mathematics right, and that previous generations were struggling with inadequate methods to reach today's understanding.

Such an attitude harms our ability to understand the great works of the past, which must be understood in their own terms.

Derision directed at Euler's definition of the derivative is a blatant example. (Euler was *not* trying unsuccessfully to describe a limit and he was not engaging in hocus pocus with differentials.)

Abel's theorem is another example.

I hope to show in this talk that the modern practice of framing Abel's theorem in terms of complex variables and Riemann surfaces not only obscures Abel's great insight and the way he approached the subject, but also leads to statements of the theorem that are less satisfactory than Abel's original.

Everyone agrees that Abel's theorem is a generalization of Euler's addition on elliptic curves, which in its simplest form states that, for a curve of the form  $y^2 = f(x)$ , where  $f(x)$  is a polynomial of degree 4 with distinct roots and  $f(0) \neq 0$ , the equation

$$\int_0^p \frac{dx}{y} + \int_0^q \frac{dx}{y} = \int_0^r \frac{dx}{y}$$

which is on its face a *transcendental* relation among the upper limits of integration  $p$ ,  $q$ , and  $r$ , is in fact an *algebraic* relation among them. (The integrals are to be taken along the curve  $y^2 = f(x)$  and  $p$ ,  $q$ , and  $r$  are assumed to be near enough to zero that the denominator  $y$  is not zero.)

Specifically, the algebraic relation among them is such that both coordinates of the point on the curve where  $x = r$  can be expressed *rationally* in terms of the coordinates of the points where  $x = p$  and  $x = q$ .

An astonishing discovery that led to much further study.

This statement extends immediately to sums of three or more integrals. For example,

$$\int_0^p \frac{dx}{y} + \int_0^q \frac{dx}{y} + \int_0^r \frac{dx}{y} = \int_0^s \frac{dx}{y}$$

where the coordinates of the point on the curve  $y^2 = f(x)$  where  $x = s$  can be expressed rationally in terms of the coordinates of the points where  $x = p$ ,  $q$ , and  $r$ .

The differential  $\frac{dx}{y}$  on the curve  $y^2 = f(x)$  in fact has *no poles* on the curve—that is, it is what is called a *differential of the first kind* or a *holomorphic differential* on the curve.

[ $2y dy = f'(x) dx$ , so  $\frac{dy}{f'(x)} = \frac{dx}{y}$ . When  $y = 0$ ,  $f'(x) \neq 0$  by virtue of the assumption that  $f(x)$  has distinct roots.]

This means that an integral of  $\frac{dx}{y}$  along the curve is meaningful even when the interval of integration includes a point where  $y = 0$ , but, when it does,  $y$  rather than  $x$  needs to be used as the parameter of integration.

Therefore, a more coordinate-free way to write the formula is

$$\int_O^P \frac{dx}{y} + \int_O^Q \frac{dx}{y} + \int_O^R \frac{dx}{y} = \int_O^S \frac{dx}{y}$$

where  $O$ ,  $P$ ,  $Q$ ,  $R$ , and  $S$  are points on the curve  $y^2 = f(x)$  and, for the sake of simplicity, they are assumed to be fairly close together, so all integrals can be computed using the same parameter. The wonderful fact is that the coordinates of  $S$  can be expressed *rationally* in terms of the coordinates of  $O$ ,  $P$ ,  $Q$ , and  $R$ .

Now forget about the right hand side of the equation and consider instead the equation

$$\int_O^P \frac{dx}{y} + \int_O^Q \frac{dx}{y} + \int_O^R \frac{dx}{y} = \text{const.}$$

The sum of the integrals on the left will remain constant as long as the expressions of the coordinates of  $S$  in terms of those of  $O$ ,  $P$ ,  $Q$ , and  $R$  remain constant. This implies an algebraic condition on the coefficients of these points.

Specifically, if either coordinate of  $S$  remains constant, so does the other, so the rational function of the coordinates of  $O$ ,  $P$ ,  $Q$ , and  $R$  that gives this coordinate must remain constant, which implies a algebraic relation  $\mathcal{R}(O, P, Q, R) = 0$  that suffices to imply that the sum of the integrals remains constant as the upper limits of integration vary.

But clearly  $O$  is irrelevant in

$$\int_O^P \frac{dx}{y} + \int_O^Q \frac{dx}{y} + \int_O^R \frac{dx}{y} = \text{const.}$$

If this quantity is constant for one  $O$  it is constant for any other  $O$ . In other words, one is talking about *indefinite integrals* evaluated at  $P$ ,  $Q$ , and  $R$ . Seen in this way, the statement is that *the sum of indefinite integrals*

$$\int \frac{dx_1}{y_1} + \int \frac{dx_2}{y_2} + \int \frac{dx_3}{y_3}$$

or, better,

$$\int \frac{dx_1}{y_1} + \int \frac{dx_2}{y_2} + \cdots + \int \frac{dx_k}{y_k}$$

is constant when the coordinates  $x_1, y_1, x_2, y_2, \dots, x_k, y_k$  satisfy a suitable algebraic relation.



Finally, consider what happens when the holomorphic differential  $\frac{dx}{y}$  is replaced by an arbitrary differential  $\omega(x, y) dx$  on the curve. Euler—we are still in the 18th century here—showed that his formula applied in this more general case as well, but the sum of the integrals

$$\int \omega(x_1, y_1) dx_1 + \int \omega(x_2, y_2) dx_2 + \cdots + \int \omega(x_k, y_k) dx_k$$

while it is no longer constant, *can be expressed in terms of elementary functions of the coordinates*  $x_1, y_1, x_2, y_2, \dots, x_k, y_k$  when these coordinates are constrained to satisfy a certain algebraic relation—the same algebraic relation for all differentials.

(Abel doesn't say “elementary functions” he says “algebraic and logarithmic functions.”)

Concluding form of Euler's addition: If  $k$  points  $(x_i, y_i)$  move along a curve  $y^2 = f(x)$ , there is an algebraic constraint on their motion with the property that

$$\int \omega(x_1, y_1) dx_1 + \int \omega(x_2, y_2) dx_2 + \cdots + \int \omega(x_k, y_k) dx_k$$

can be expressed, for any differential  $\omega(x, y) dx$ , in terms of elementary functions in the coordinates  $x_1, y_1, x_2, y_2, \dots, x_k, y_k$  when these coordinates satisfy the algebraic constraint.

(This function—being an indefinite integral—is determined only up to an additive constant.)

Note the relation of this statement to the major problem confronted in the early years of the study of calculus, namely, the problem of *integration*. Given a function of one variable, when can you find a function of which it is the derivative?

This would necessarily mean, when can you find a function you *already know about* of which it is the derivative? For Abel, those functions were algebraic and logarithmic functions.

The integral of  $\omega(x, \sqrt{f(x)})$  *can not* be expressed in terms of elementary functions but a particular sum of such integrals *can* be so expressed when the moving points move in such a way that their coordinates satisfy a certain algebraic relation.

In his famous Paris memoir, Abel states Euler's conclusion in almost exactly this form—taking it to be a known fact—as a preamble to his theorem. His theorem, very slightly paraphrased, is then:

*The same thing is true for any plane curve  $\chi(x,y) = 0$ , except that you will normally need to impose more than one algebraic condition on the motion of the points.*

More fully: *A set of  $k$  points  $(x_i, y_i)$  moving along a curve  $\chi(x, y) = 0$  can be subjected to a finite number of algebraic constraints in such a way that a sum of indefinite integrals*

$$\int \omega(x_1, y_1) dx_1 + \int \omega(x_2, y_2) dx_2 + \cdots + \int \omega(x_k, y_k) dx_k$$

*can be expressed in terms of elementary functions of the coordinates of the moving points provided these coordinates satisfy the constraints.*

The constraints are independent of  $\omega(x, y) dx$ .

Note the emphasis on “algebraic.” It is Abel’s.

The number  $g$  of conditions that are needed depends only on the curve. The minimum number of conditions that suffice for a given curve  $\chi(x, y) = 0$  is, as you may have guessed, the **genus** of the curve.

Note: No complex variable, no Riemann surfaces.

It is not remarkable, after all, that Abel was able to discover his theorem without them. In his formulation of the problem they simply did not arise.

I don't find this simple form of the theorem in the literature (other than in Abel's memoir itself). The reason, I believe, is that Abel's setting of the theorem does not fit well with modern ways of thinking.

For one thing, the overriding importance of the function concept in modern mathematics obscures the theorem. Abel dealt constantly with "algebraic functions," which are not functions at all in the modern sense (they are "many valued") and the statement of his theorem involves an indefinite integral, which is also not a function in the modern sense (it is determined only up to an additive constant).

My own experience was that I had to remind myself many times that

$$\int \omega(x_1, y_1) dx_1 + \int \omega(x_2, y_2) dx_2 + \cdots + \int \omega(x_k, y_k) dx_k$$

depends on  $k$  moving points (that's what the  $dx_i$ 's indicate) and is not a sum of  $k$  values of the same function.

In the 20th century, most statements of Abel's theorem have derived from the statement given by Hermann Weyl in his famous 1913 book *Die Idee der Riemannschen Fläche*. He in fact made two drastic revisions.

On the one hand, he extended it, making it in the process into a theorem that depended on the topology of the Riemann surface to which the curve corresponds. As Weyl admits, this part of his theorem is far from anything in Abel, although he continued to call it Abel's theorem.

On the other hand, he stated the part that was from Abel in a way that in effect described the  $g$  constraints on the motions of  $k$  points in terms of a *differential equation*, namely, the differential equation determined by the holomorphic differentials, rather than the *algebraic* constraints that Abel insisted on.



Abel's emphasis on *algebraic* constraints caused me to be dissatisfied with *differential* constraints and led me eventually to a statement and proof of the theorem that relegated the  $k$  moving points to the status of a heuristic device. Instead, both theorem and proof can be expressed entirely in terms of the algebra of polynomials with integer coefficients.

It is not surprising that such a statement and proof of the theorem is possible, because these were the techniques Abel used. He did differentiate and integrate, of course, but he worked always with algebraic functions and algebraic quantities, so his differentiation and integration inevitably were, in the last analysis, algebraic operations.

All of this is perhaps a bit technical for a history talk, but the concepts are all from Calculus 1, and the main point I want to draw from it is a point about the study of history.

The primary goal of my study of history of mathematics—others assuredly have had other motives—has always been to understand the classic works of the great mathematicians and to make those works available to today's readers.

Useful as our modern formulations can often be, those great works can be truly understood only in their original context and only on their own terms.