

Style and Rigor in Mathematics

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One of the primary goals of my work in the history of mathematics is to make known the way in which Leopold Kronecker's effort to base the mathematics of his and preceding generations on what he called "generalized arithmetic"—the algebra of polynomials with integer coefficients—was overruled at the end of the 19th century and never revived. This is a question of "style" in a very broad sense. Kronecker's style of algorithmic and finitistic mathematics, which bases concepts and proofs on concrete polynomial constructions, satisfies my own demands, both aesthetic and technical, but is so antithetical to today's transfinite set-theoretic constructions that it is rejected as unworkable today. The prevailing belief is that there is only one rigorous way to do mathematics and that it must be followed. My thesis is that mathematics would be enriched by opening the forum to other styles of thought and presentation.

I was encouraged by James Pierpont's statement in his essay "Mathematical Rigor, Past and Present," [5] that "Personally [I] do not believe that absolute rigor will ever be attained and if a time arrives when this is thought to be the case, it will be a sign that the race of mathematicians has declined." Pierpont, after making this statement, so surprising to today's mathematicians, goes on, not to attempt to describe rigor, but to "pass in review some examples of what were regarded at the time as good mathematical demonstrations."

My article "Euler's Definition of the Derivative" [2] presents a view of Euler's standards of rigor that is very different from Pierpont's, who states that, "Judged by modern standards [Euler's] demonstrations are quite worthless." I believe that, carefully read and properly understood, Euler's demonstrations are as rigorous and convincing as modern mathematics, as I tried to show in the specific case of Euler's treatment of derivatives. It is rejected today for reasons of style, not of rigor. The modern reader tends to believe Euler is describing limits in an inadequate way, but in fact Euler's definition of the derivative does not involve limits at all.

A key attitude of the second half of the 19th century was expressed by Richard Dedekind when he said [1]: “My efforts in number theory have been directed toward basing the work not on arbitrary representations [*Darstellungsformen*] or expressions but on simple foundational concepts and thereby—although the comparison may sound a bit grandiose—to achieve in number theory something analogous to what Riemann achieved in function theory, in which connection I cannot suppress the passing remark that in my opinion Riemann’s principles are not being adhered to in a significant way by most writers—for example, even in the newest works on elliptic functions; almost always they disfigure the theory by unnecessarily bringing in forms of representation [*Darstellungsformen* again] which should be results, not tools, of the theory.” (My translation.)

I summarized the argument I made against this statement of “Riemann’s principles” in a talk I recently gave [3] with the title “The Algorithmic Side of Riemann’s Mathematics.” Invoking Riemann’s work on the Riemann-Siegel formula, on establishing the analytic continuation and the functional equation of the zeta function, on transforming hypergeometric functions, and on conceptualizing and working with “Riemann surfaces,” I tried to show that Riemann was not only a master of what Dedekind called *Darstellungsformen*, but also that they were very much tools, not results, of his theories.

Dedekind’s attitude is repeated and even amplified in David Hilbert’s statement in the introduction to his famous *Zahlbericht* [4] that, “I have sought to avoid Kummer’s vast computational apparatus and thereby to realize Riemann’s fundamental principle that proofs should be effected not by computation but solely by concepts” (my translation). To me, this approach to the subject not only deprives his readers of the experience of Kummer’s fertile and beautiful techniques but is a degradation of their rigor insofar as Hilbert replaces the banned “computational apparatus” with “constructions” that are transfinite algorithms that fall far short of what Kummer and his student Kronecker would have regarded as rigor.

Kronecker’s ideas of rigor are indicated in a famous statement, “If I still have the time and the energy, I will myself show the mathematical world that not only geometry but also arithmetic can point the path to analysis, and certainly a more rigorous one. If I cannot do this, then another will who comes after me, and the world will recognize the inexactitude of

the types of proof now employed in analysis” (my translation). This and other indications Kronecker gave, as well as the body of his mathematical work, show clearly that he wanted to base all of his concepts and proofs on finite (but not necessarily practical) algorithms and computations. Pierpont shows sympathy and even admiration for Kronecker’s view.

I differ from Pierpont, however, when he casts L. E. J. Brouwer as the mathematician of Pierpont’s time whose principles were closest to Kronecker’s. From the point of view of style, Kronecker and Brouwer could hardly be more different. Kronecker was a product of a classical German *Bildung* while Brouwer was a mystic. Kronecker was primarily interested in mathematics, not the philosophy of mathematics, and he was a careful student of both the classics of mathematics and the work of his contemporaries, while Brouwer worked in mathematics primarily to validate his philosophical principles, and worked in the new field of topology in idiosyncratic ways. Kronecker was a banker, while Brouwer was a prophet.

In my opinion, the association of Brouwer with Kronecker’s program has done great damage to a proper understanding of what Kronecker’s vision for mathematics was.

In conclusion, I believe that a broad exploration of various styles—in the sense of the word I have tried to indicate—would enrich mathematics and promote rigor in the only way that remains possible if one agrees with Pierpont that absolute rigor is a mirage. It would release the stranglehold that set theory currently has on mathematics and promote approaches to topics like number theory, algebraic geometry, and the classical theory of functions that are better adapted to these topics and that use more constructive, direct, and comprehensible methods.

REFERENCES

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