Euler's Conception of the Derivative

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Euler begins the preface to volume one of his multivolume work on the differential and integral calculus[1] with the statement:

> Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art. From this it follows not only that they remain on the fringes, but in addition they entertain strange ideas about the concept of the infinite, which they must try to use.[2]

This describes perfectly the situation of the authors of the preceding article. They are attempting to understand *how* Euler uses differentials in later volumes without having understood his explanation in the first volume of what he is using differentials *for.* As a result, they have, as Euler foretold, come to entertain strange ideas about the concept of the infinite. Their strange ideas derive, however, from 20th century notions of set theory and nonstandard analysis that are far stranger than anything Euler could have imagined.

The authors cite H. J. M. Bos's article *Differentials and Derivatives in Leibniz's Calculus* [Bos 1974], but they appear to have read it no more carefully than they read mine [Edwards 2007], because Bos's descriptions of Euler's ideas concur with those in my paper, not with those being advanced by the authors.

Bos writes on page 66:

> The aim of [Euler's] arguments is to establish that, although the concept of the infinitely small cannot be rigorously upheld, still the computational practice with differentials leads to correct results. ... Euler claimed that infinitely small quantities are equal to zero, but that two quantities, both equal to zero, can have a determined ratio. This ratio of zeros was the real subject-matter of the differential calculus ...

He goes on to say that "Euler also considered this ratio of zeros as a limit" and quotes (p. 67) a statement from Euler's *Institutiones Calculi Differentialis* describing the limit in a way that fully substantiates my claim that "*of course* Euler understood limits." Bos then quotes another passage in which Euler explains his understanding of the meaning of $\frac{dy}{dx}$:

> Although the rules, as they are usually presented, seem to concern evanescent increments, which have to be defined; still conclusions are never drawn from a consideration of the increments separately, but always their ratio .... But in order to comprise and represent these reasonings in calculations more easily, the evanescent increments are denoted by certain symbols, although they are nothing; and since these symbols are used, there is no reason why certain names should not be given to them. [Bos's translation. See his footnote 110 for the original Latin.]

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[1] It consists of six volumes published over 20 years. The first two volumes *Introductio in Analysin Infinitorum* (Introduction to the Analysis of Infinites) were both published in 1748, the third *Institutiones Calculi Differentialis* (Foundations of Differential Calculus) was published in 1755, and the last three *Institutiones Calculi Integralis* (Foundations of Integral Calculus) were published in 1768, 1769, and 1770.

[2] As translated by J. Blanton. The original Latin is quoted in [Edwards 2007], footnote 4.

True, Euler gives no definition of the derivative, and the title "Euler's Understanding of the Meaning of $\frac{dy}{dx}$" would have been more descriptive of my paper, but I think the title I used conveys the idea pretty well to modern readers unless they expect Euler to have written in a modern definition-theorem-proof style.

My recipe "rewrite $\frac{\Delta y}{\Delta x}$ in a way that still makes sense when $\Delta x = 0$", which failed to convince the authors, is, I maintain, the essence of Euler's method for *evaluating* limits. It is based on the tacit assumption that any two such ways of rewriting $\frac{\Delta y}{\Delta x}$ will give the same value when $\Delta x = 0$. This is a far less adventuresome assumption than Euler's analogous assumption that any two methods (within reason) of summing a divergent series will result in the same sum. (On this fascinating topic, see [Hardy 1949].)

I thank the authors for their mention of "Edwards curves," especially since I see no possible connection with the subject of their article. I would point out, though, that their form $x^2 + y^2 = 1 + dx^2y^2$ is the one used in cryptography [Hisil et al. 2009], not the one that I proposed (which was $x^2 + y^2 = a^2 + a^2x^2y^2$), and, in the context of their article about higher order differentials, readers may well think that dx^2y^2 is some weird kind of differential.

The authors misconstrue my meaning when I say in my footnote 5 that students today are taught to shun differentials as they would an infectious disease. I am agreeing with Euler that students *should* learn the calculus of differentials, not agreeing with Cantor that infinitesimals are the cholera bacillus of mathematics or with the orthodox view in modern times (before the advent of nonstandard analysis) that differentials are for physicists and applied mathematicians who don't care about correct mathematical reasoning.

One last note: I regret the authors' inference that my failure to name Lagrange in one of my informal lists of "masters" means that I do not hold him to be a master. Nothing could be farther from the truth. In my opinion, the technique of Lagrange multipliers is one of the great ideas in the history of mathematics—I emphasize it in my book *Advanced Calculus*—and at the beginning of my book *Galois Theory* I hypothesize that Lagrange in effect became the master of the pupil Galois when Galois, early in his studies, read Lagrange's great work on the algebraic solution of equations. And these are but two of Lagrange's many contributions.

References

[Bos 1974] Bos, H. J. M., *Differentials, Higher Order Differentials, and the Derivative in the Leibnizian Calculus,* Arch. Hist. Exact Sci., 14 (1974) 1-90.

[Edwards 2007] Edwards, H. M., *Euler's Definition of the Derivative,* Bull. AMS, 44 (2007) 575-580.

[Edwards 1969] Edwards, H. M., *Advanced Calculus*, Houghton Mifflin 1969 (Republished by Krieger in 1980 and by Birkhäuser in 1994).

[Edwards 1984] Edwards, H. M. *Galois Theory,* Springer-Verlag, 1984.

[Euler 1748] Euler, L., *Introductio in Analysin Infinitorum,* St. Petersburg, 1748. *Introduction to the Analysis of the Infinite,"* John D. Blanton, trans., Springer, New York, 1988.

[Euler 1755] Euler, L., *Institutiones Calculi Differentialis,* St. Petersburg, 1755. *Foundations of Differential Calculus,* John D. Blanton, trans., Springer, New York, 2000.

[Hardy 1949] Hardy, G. H., *Divergent Series,* Oxford, 1949. Chelsea reprint, 1991.

[Hisil et al. 2009] Hisil, H., Wong, K., Carter, G.,Dawson, E., *Twisted Edwards Curves Revisited,* Conferences in Research and Practice in Information Technology, vol. 98, pp. 7–19, ACS, 2009. Web address: http://iacr.org/archive/asiacrypt2008/53500329/53500329.pdf

[Kanovei et al. 2015] Euler's Lute and Edwards's Oud (this magazine).