The Use of Infinity in Mathematics

The obvious place to start a talk about infinity in mathematics is with Zeno's famous paradoxes and with Aristotle's response to them.

I'll discuss just one of Zeno's four paradoxes, the old favorite about Achilles and the tortoise. The tortoise has a head start in a footrace with Achilles. Zeno argues that the swift Achilles can never catch up, because before he can overtake the tortoise he must reach the tortoise's starting point, by which time the tortoise will have moved on. So he must then cover the distance that the tortoise has moved, by which time the tortoise will have moved even farther, and so forth. Achilles has to make an infinite number of steps of this type which, when you look at it this way, seems impossible. How can you do an infinite number of steps in finite time?

The problem here is a ridiculously abstract one. One can't doubt for a moment that Achilles will overtake the tortoise. The problem consists of trying to think of it as happening as a result of an infinite number of steps.

The sane answer, it seems to me, is the one Aristotle gave. Don't. Don't try to think of it as occurring in an infinite number of steps. You can perhaps—with a lot of effort—conceive of a million such steps, each requiring a finite time, but the last ten thousand of the first million steps are almost inconceivably quick—all ten thousand of them taken together are inconceivably quick, not to mention each step individually. Is it not fruitless nonsense to try to imagine them as the mere beginning of an infinite number of steps?

In short, avoid attempts to deal with *infinites,* either things that are infinitely small or infinitely large. Often it is useful to deal with *potential* infinites—for example to deal with the sequence of numbers 1, 2, 3, ... not as an actually infinite sequence but as a sequence that can be prolonged for as long as you like. Its *potential* is infinite in the sense that it never reaches termination, but it is not *actually* in itself infinite.

To give another example of potential as opposed to completed or actual infinites, consider that modern technology routinely deals with nanoseconds—a nanosecond is one billionth of a second, in case you have forgotten—which a century ago would have been considered to be virtually "infinitesimal." Small as a nanosecond is, we can certainly think of partitioning it further. There is always the potential of going farther. In some scientific fields I'm sure this is done. But it is not within the realm of science to regard the process as ever being *completed* so that the nanosecond is partitioned into an infinite number of infinitesimal time intervals. (By the way, all but the first dozen or so steps Achilles must make to overtake the tortoise are tiny fractions of a nanosecond.)

Aristotle said: "Nor does this account of infinity rob the mathematicians of their study; for all that it denies is the actual existence of anything so great that you can never get to the end of it. And as a matter of fact, mathematicians never ask for or introduce an infinite magnitude; they only claim that the finite line shall be of any length they please; and it is possible to divide any magnitude whatsoever in the same proportion as the greatest magnitude."

The last part refers to the fact that you can divide magnitudes as finely as you please—but not, of course, infinitely finely.

One certainly can't say that Aristotle put a stop to speculations about infinity—that will never happen—but I think you can say that he made them unrespectable in many important venues.

My topic is the use of infinity in mathematics, not philosophy, and I have nothing to say about the view of infinity adopted by the scholastic philosophers of the middle ages or by anyone else between Aristotle and Newton. Like everyone else, I have heard about disputes as to how many angels can dance on the head of a pin, but I have also heard that that presents a very inaccurate picture of the medieval thinking. I do know that Georg Cantor, about whom I will speak later, found ecclesiastical writers of the medieval period who had views of infinity that he believed were sympathetic to his own; I don't imagine they involved angels dancing on pins, but in my opinion these speculations belong to philosophy, not mathematics, and as such are not relevant to my talk.

The usual version of the history of mathematics has it that the invention of the calculus by Newton and Leibniz in the late 17th century brought infinity into mathematics where it has remained ever since. But—the standard version goes on to say—infinity is very tricky to deal with, and for the first century and a half, until the middle of the 19th century, mathematicians didn't really know how to deal with it in a satisfactory way. As mathematicians like to say, the calculus wasn't "rigorous." Then a German mathematician named Karl Weierstrass (the French mathematician Augustin Cauchy is another who is frequently credited)

presented a method—universally described in terms of the Greek letters epsilon and delta—that made possible the rigorous presentation of the differential and integral calculus that until that time had been insecure. Further work in the second half of the 19th century, mainly on the part of the German mathematician Georg Cantor, brought to mathematics a rigorous theory of infinite magnitudes— *pace* Aristotle—that was almost universally accepted by the beginning of the 20th century and overcame most of the problems associated with infinity.

Well, even in the standard version, a few doubters remained. Sometimes the standard version acknowledges that the eminent French mathematician Henri Poincare criticized set theory severely. In Germany, David Hilbert was the great enthusiast for the Cantorian revolution, but his favorite student, Hermann Weyl, flirted for many years with the theories of the Dutch opponent of Cantor, L. E. J. Brouwer. By the time I was in graduate school in the 1950's, Brouwer and his followers had been thoroughly marginalized and Cantor and Hilbert had won out.

In the 1950's there was, admittedly, a great deal of talk about problems on the frontier of the theory of infinite sets—you may have heard of Russell's paradox, the axiom of choice, the continuum hypothesis and other questions which it is hard for me to think of without remembering those dancing angels. But the establishment view was certainly that these recondite problems were the result of having pushed the frontiers of mathematics to the very farthest point. Everyone (except a few kooks) believed that the correct handling of the infinitesimals involved in the calculus was a fait accompli.

Since I propose this version of history only to attack it, you should be asking whether I am setting up a straw man, whether this really is the usual version. For verification I refer you to Morris Kline's compendium *Mathematical Thought from Ancient to Modern Times*, a very scholarly book, or Tobias Dantzig's popular book *Number, the Language of Science.* Another reference would be Judith Grabiner's valuable book *The Origins of Cauchy's Rigorous Calculus.* Some of you may be able to provide me with even more references.

My version of the story is quite different. I believe that if you read Newton's description of a limit, you will see the modern epsilon and delta definition, although of course he doesn't use that terminology. Bishop Berkeley's witticism about the numerator and denominator of the derivative being "ghosts of departed quantities" is a good joke, and it scored the theological points he wanted to make, but it relates not to the ideas of the great Newton but to the difficulties ordinary minds had in following the great Newton.

A Newton scholar recently told me that Newton's great work, the *Principia,* repeatedly uses the phrase *quam proxime,* meaning *very, very nearly* to describe the relation between theoretical results and computations or experimental data. To think carefully about approximations and experimental error—and no one in the history of the world has thought more deeply about such matters than Newton—is to understand the meaning of limits.

I don't know whether Newton would have felt that to heed Aristotle would have "robbed the mathematicians of their study," but I doubt it. In any case, this is a subjective question that even a careful reading of Newton's voluminous

writings could never answer. I would argue, however, that Newton's understanding of the limit concept could not possibly have been deepened by studying the works of Weierstrass or Cauchy. His reaction would surely have been, "Well, yes, of course."

Newton's rival Leibniz is more frequently accused of dallying with infinitesimals, perhaps because his marvelously suggestive and effective notation for calculus might seem to careless thinkers to deal in infinitesimals, but S. F. Lacroix had this to say on the subject of Leibniz and differentials in the introduction to his treatise on the calculus published in 1797. (The alternative year of publication was the year 5):

"Leibniz seems to have believed that those who were able to use the differential calculus would easily grasp its spirit, by comparing it with the method of the Ancients, because he neglected to enter into any detail whatsoever in this regard, and his silence was imitated by Bernoulli and L'Hopital; but when he was attacked on this subject, he showed by his responses that he had reflected maturely on it. On all occasions, he compares his method with that of Archimedes, and makes clear that his is nothing but a sort of abbreviation of the other, more appropriate to research, but in the end it amounts to the same thing; because instead of supposing the differentials actually to be infinitely small, it suffices merely to conceive that one can always make them so small that the error one commits by omitting them in the calculation is less than any given magnitude."

Forgive me for quoting Lacroix at such length, but in essence his statement is almost exactly the same statement I want to make today, two hundred and

seven years later. The great mathematicians—at least until the late nineteenth century—did not deal with completed infinites in the way that lesser minds have often interpreted them as having done. They had reflected maturely on it and understood that, in Lacroix's words, "it suffices merely to conceive that one can always make [the quantities] so small that the error one commits by omitting them in the calculation is less than any given magnitude." For those of you acquainted with the epsilon-delta definition, let me point out that the "any given magnitude" is epsilon and "making the quantities so small" is delta.

I have not made a close study of Leibniz's writings, so I can't say that Lacroix's description of them is justified. I believe it very likely is, but in any case Lacroix himself clearly had reflected maturely on the foundations of calculus and had a good grasp of the idea of a limit well before the time of Weierstrass and Cauchy.

I also have to share with you what Lacroix says next, because it is such a wonderful commentary on the established opinion about just about anything.

"This method of reasoning, which would seem to be beyond reproach, was seen by Fontenelle as a confession on Leibniz's part of the inadequacy of his principles, from which would follow the collapse of the entire edifice he had erected on infinites. The complaints Fontenelle makes about it in the preface to his geometry, and which have been repeated in many works, offer an example of the ease with which errors pass from book to book, and shows how few people take the trouble to form an opinion independent of that of others."

That statement should be put on a plaque and posted in educational institutions everywhere. And in 2004 one must add that, however easily errors passed

from book to book in 1797, they pass far more easily from website to website in 2004.

The great mathematician Carl Friedrich Gauss, who lived from 1777 to 1855—a century and more after Newton and Leibniz—was conserving the Aristotelian tradition when he said that completed (*Vollendeten*) infinites are never allowed in mathematics and that infinity should *always* be understood as a mere "*façon de parler*."

How did it happen that a century that began with the esteemed Gauss subscribing to the Aristotelian view ended with the triumph of Cantor's views, and the esteemed David Hilbert exploring the outer fringes of the theory of the infinite?

That is of course a complex story involving many ideas and many mathematicians, but let me oversimplify it by saying that it was all the doing of Fourier and Riemann, in that order.

Joseph Fourier, whose dates are 1768-1830, was a French mathematician, physicist, and politician, who in a certain sense was not a mathematician at all. He is most famous for his theory of heat, and is thought of as a pioneer not in mathematics but in mathematical physics. There is an old joke that applied mathematics is to mathematics as military music is to music. I don't agree with the implication that applied mathematics has none of the beauty or elegance of pure mathematics, but I do agree with the implication that it isn't really mathematics.

Mathematics is in its essence deductive, but applied mathematics is in its essence inductive. Fourier's great discovery was a set of mathematical techniques, known today as Fourier analysis, that made it possible to describe certain physical

phenomena in a way that was doubly breath-taking, breath-taking for its sweep and elegance at the same time that it was breath-taking for its audacity in abandoning any pretense of solid mathematical underpinning.

A theory like Fourier's is the best thing that can happen to mathematics. It is to mathematics what the discovery of an unexpected phenomenon is to physics. It forces a drastic re-thinking of basic principles in order to bring them into agreement with the new point of view and the new data.

Much mathematical work in the 19th century was indeed devoted to taming and absorbing Fourier analysis into the body of mathematics, and much beautiful mathematics was created in that way. But the tension between the great generality of the results of Fourier analysis and the demands of deductive mathematics were not fully resolved.

Bernhard Riemann, a German whose amazing career was cut short by tuberculosis at the age of 40 in 1866, was like Fourier intensely interested in mathematical physics, but unlike Fourier no one would dare say of him that he was not really a mathematician. He was one of the greatest mathematicians of all time, but in a unique way. His ideas were astonishingly original, and, like Fourier's, they often left mathematical justifications far behind, but they were based on a mathematical intuition and a mastery of mathematical technique that were unsurpassed.

Let me tell you a highly condensed version of the story of the famous Riemann hypothesis. Riemann remarked in a paper on the subject of the distribution of prime numbers that he thought it "very likely" that the zeros of a certain complicated function which he described in terms of a very specific definite integral

are all on the real axis. That opinion is now called the "Riemann hypothesis." Today, 145 years after Riemann's paper was published, the mathematicians of the world agree that the Riemann hypothesis is very likely true, but to *prove* it is true is universally acknowledged to be the greatest unsolved problem in mathematics today.

It is marvelous enough that Riemann made this fundamental discovery, but the really marvelous part of the story is yet to come. By 1930, the importance and the difficulty of the Riemann hypothesis was well understood, but I think the general impression was that it had been a kind of a lucky guess on Riemann's part. However, when the mathematician and scholar Carl Ludwig Siegel undertook to analyze Riemann's unpublished papers in the Göttingen archives, he found that Riemann had done numerical calculations of the first few zeros of the function in question using techniques more powerful than any that had been found by other mathematicians in the intervening 70 years. When Siegel published these techniques of Riemann in a paper that was part history and part mathematics, the notion that Riemann's hypothesis was a lucky guess was laid to rest and the range of his explorations began to seem almost super-human.

Siegel expressed his estimate of Riemann in the following way: "It is not as widely believed today [as it had been a few decades before] that Riemann reached his conclusions by means of grand general ideas, but few have realized how strong Riemann's technique truly was."

What does this have to do with Aristotle and the infinite? Everything. As with Newton and Leibniz, those who came after Riemann were unable to pursue his

ideas without his masterful insights, and their attempts to base Riemann's work on a solid foundation and to apply his modes of thinking in other circumstances were seriously flawed.

Which brings me back to Weierstrass and Cantor and another German mathematician of the period named Richard Dedekind, who lived from 1831 to 1916 (and who, by the way, was a personal friend of Riemann). It is generally agreed, and for once I agree as well, that these men wrought a revolution in thinking about the foundations of mathematics, principally in attitudes toward the infinite. You may well have heard of the notion of a "Dedekind cut," which is Dedekind's version of the definition of a "real number." On the one hand, the idea of a Dedekind cut brushes aside Aristotle's (and Gauss's) objection to completed infinites. On the other hand it is widely accepted today as the correct and rigorous way to define real numbers.

The sources I referred to above, and countless others, tell the story of this revolution and describe it as a triumph of the human spirit. Though I admire much in Dedekind's work, I believe his interpretation of Riemann's work and his notion of the Dedekind cut were not positive contributions.

And, finally, we come now to the last of the mathematicians I will discuss, Leopold Kronecker. Kronecker, who lived from 1823 to 1891, was one of the few nay-sayers during the Weierstrass-Cantor-Dedekind revolution. His message was: *Hold on! This drastic revision of the principles of our subject is not necessary!*

He had few allies—though he was one of the most highly regarded mathematicians of his generation—and it must be conceded that he lost the fight.

The successes of Fourier and Riemann and others seem to have convinced mathematicians of the time that a new era had dawned, that the conservative attitudes of Aristotle and Gauss could be disregarded because more daring attitudes toward reasoning with infinites produced reliable results.

In Kronecker's day the main issue related to real numbers. Kronecker, true to the classical tradition, realized that there was no way to conceive of a general real number except as a completed infinite.

In simple terms, a real number is an infinite decimal expansion, such as the familiar expansion 3.14159... of π. Now π can be described in other ways—for example, it is 4 times 1 - (1/3) + (1/5) - (1/7) + (1/9) - .... At first glance, this second description may seem to have little to recommend it over the first. It, too, is an infinite process. Worse, you would have to compute a staggering number of terms of the alternating sum to achieve the five place accuracy given in the decimal expansion. But the difference is that the alternating sum tells you exactly how to get 10 place accuracy in the decimal expansion (even though the computation it specifies is utterly unworkable) whereas the three dots after 3.14159... give you no idea whatever. Do you recognize the difference? The second description is a *potential* infinite—it tells you how to compute π as accurately as you like but not how to compute it infinitely accurately—whereas the the first implies the notion of π as an *actual* and *exact* value—a thing that *is,* not one that *becomes.*

Kronecker maintained that the rush to accept completed infinites was unnecessary and undertook in his own work to show that it was as true in his own day

as it had been in Aristotle's that the exclusion of completed infinites did not rob the mathematicians of their study.

But the trend of the times was against him. The other mathematical leaders of the time were ready to accept not only the completed infinite that is π, but the immeasurably greater completed infinite that is *the set of all real numbers,* of which π is a humble element among uncountably many (in a technical Cantorian sense) others.

In accounting for the success of this revolution in attitudes toward infinity one must surely acknowledge not only the works of Fourier and Riemann but also those of Cantor.

For those of you who have been initiated into the mystery of the Cantorian theory, let me mention Cantor's fascinating two part proof of the existence of transcendental numbers; he established first that the set of all real numbers was uncountable and then that the set of all algebraic numbers was countable, leading to the conclusion that the set of transcendental numbers—those that are real but not algebraic—is not only nonempty but is in fact uncountable.

There is an irony here. Cantor's proof can be made to conform perfectly well to Kronecker's principles and to *construct* a specific transcendental number, a real number that is not a root of any algebraic equation. But the very statement of the theorem—that transcendental numbers exist—is so incompatible with Kronecker's idea of mathematics that there seems to be little point in going through such a construction.

Kronecker said we can do without "the set of all real numbers" and he preferred to do without it because he felt that letting in completed infinites risked letting in ambiguities, doubts, and complexities that would discredit mathematics and reduce its appeal and its authority.

And, most importantly, he maintained that the admission of completed infinites *in no way benefitted mathematics.*

But, Kronecker to the contrary notwithstanding, the years following his death were filled with discussions of completed infinites—issues I mentioned before involving Russell's paradox, the continuum hypothesis, the axiom of choice, the Zermelo-Frankel axioms for set theory, and so forth. Studies of this sort came to be accepted as being about "the foundations of mathematics" even though Kronecker would not have regarded them as being about mathematics at all, much less about the foundations of mathematics.

Infinity proved addictive. Once the habit of using completed infinites in mathematical arguments has become established, it is hard to root out. It is part and parcel of mathematics today, and Kronecker is regarded as ridiculously old-fashioned and reactionary by all but a small minority of us today.

But he is on the upswing. And he is on the upswing for a clear and persuasive reason. Computers.

Computers have changed the way that mathematicians think, and they have done so in a way that opposes the acceptance of completed infinites. Computers compute. They demand *algorithms,* schemes of computation, and that means schemes that arrive at *results* not only at the end of finitely many steps but even

at the end of a number of steps that can be accomplished in a reasonable amount of time. When you are thinking in terms of algorithms, you must look askance at a procedure that promises to produce an answer eventually but makes no promise to do so in less than a hundred million years. So the alternating series for π I mentioned before is laughable as a method of computing π, but when you are thinking algorithmically it seems much less laughable than those silly three dots following 3.14159 that appear to say something but in fact say nothing.

Riemann's successors, lacking Riemann's technical power, focussed on the aspect of Riemann's work that was most accessible and most admirable to them, which was his ability to pose problems in extreme generality and abstractness, free to the largest degree possible of specific formulas and computations. But modern mathematicians with computers at their disposal can follow up on other aspects of Riemann's work. The techniques that Siegel found Riemann had used to compute the first few zeros of his mysterious function by hand have been used by modern mathematicians to compute the first few billion—yes, I did say billion—zeros of Riemann's function.

In this connection, it is interesting to note that another important technique used in computing zeros of Riemann's function was developed in the mid 20th century by a mathematician who was one of the principal fomenters of the computer revolution, Alan Turing. Although Turing is a thinker of great importance in the field of mathematical logic and thereby in the theory of infinite sets, he also thought very concretely in terms of actual computations and in this way participated in yet another revolution in mathematical thought.

The new era of unimaginably powerful computers will inevitably give birth to a new era of mathematics. It is impossible to guess what the shape of that new mathematics might be, but there is good reason to believe—in my case, to hope—that the 20th century addiction to completed infinites may be overcome, that the ability to execute calculations of tremendous length will make inevitable the distinction between tremendously long calculations and calculations that are never intended to terminate. It would please me very much to see Aristotle and Kronecker vindicated and to see the refusal to allow completed infinites have the result not only of *not robbing* the mathematicians of their study but of *vastly enriching* that study.