LECTURES ON THE FOURTH ORDER Q CURVATURE EQUATION

FENGBO HANG AND PAUL C. YANG

Abstract. We discuss some open problems and recent progress related to the 4th order Paneitz operator and Q curvature in dimensions other than 4.

1. Introduction

In conformal geometry, a major tool is a family of conformal covariant operators and their associated curvature invariants. In dimension $n > 2$, the conformal Laplacian operator

$$L = -\frac{4(n-1)}{n-2}\Delta + R \tag{1.1}$$

enjoys the following covariance property,

$$L_{\rho^{\frac{4}{n-2}}g}\varphi = \rho^{-\frac{n+2}{n-2}} L_g(\rho\varphi) \tag{1.2}$$

for any smooth positive function ρ (see [LP]). Here R denotes the scalar curvature. The associated transformation law of scalar curvature follows,

$$R_{\rho^{\frac{4}{n-2}}g} = L_{\rho^{\frac{4}{n-2}}g}1 = \rho^{-\frac{n+2}{n-2}} L_g\rho. \tag{1.3}$$

A fundamental result is the solution of the Yamabe problem [Au2, S, T, Y], which is related to the sharp constant of the associated Sobolev inequality. Since then, there is a large literature on the analysis and geometry of this equation. In order to gain additional information on the Ricci tensor, the 4th order Q curvature equation comes into play.

Let (M, g) be a smooth Riemannian manifold with dimension $n \geq 3$, the Q curvature is given by ([B, P])

$$\begin{aligned} Q &= -\frac{1}{2(n-1)}\Delta R - \frac{2}{(n-2)^2}|Rc|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}R^2 \\ &= -\Delta J - 2|A|^2 + \frac{n}{2}J^2. \end{aligned} \tag{1.4}$$

Here Rc is the Ricci tensor and

$$J = \frac{R}{2(n-1)}, \quad A = \frac{1}{n-2}(Rc - Jg). \tag{1.5}$$

The Paneitz operator is defined as

$$\begin{aligned} & P\varphi \\ = \ & \Delta^2\varphi + \frac{4}{n-2}\operatorname{div}(Rc(\nabla\varphi, e_i)e_i) - \frac{n^2-4n+8}{2(n-1)(n-2)}\operatorname{div}(R\nabla\varphi) + \frac{n-4}{2}Q\varphi \\ = \ & \Delta^2\varphi + \operatorname{div}(4A(\nabla\varphi, e_i)e_i - (n-2)J\nabla\varphi) + \frac{n-4}{2}Q\varphi. \end{aligned} \tag{1.6}$$

Here $e_1, \cdots, e_n$ is a local orthonormal frame with respect to g. Note that the use of J and A (Schouten tensor) simplifies the formulas of Q curvature and Paneitz operator.

In dimension $n \neq 4$, the operator satisfies

$$P_{\rho^{\frac{4}{n-4}}g}\varphi = \rho^{-\frac{n+4}{n-4}} P_g(\rho\varphi) \tag{1.7}$$

for any positive smooth function ρ. This is similar to (1.2). As a consequence we have

$$Q_{\rho^{\frac{4}{n-4}}g} = \frac{2}{n-4} P_{\rho^{\frac{4}{n-4}}g} 1 = \frac{2}{n-4} \rho^{-\frac{n+4}{n-4}} P_g \rho. \tag{1.8}$$

In dimension 4, the Paneitz operator satisfies

$$P_{e^{2w}g}\varphi = e^{-4w} P_g \varphi \tag{1.9}$$

and the Q curvature transforms as

$$Q_{e^{2w}g} = e^{-4w}(P_g w + Q_g). \tag{1.10}$$

This should be compared to the conformal invariance of $-\Delta$ on surface and the transformation law of Gaussian curvature under a conformal change of metric.

The main theme of research is to find out the role of Paneitz operator and Q curvature in understanding the geometry of a conformal class and the topology of underlying manifold. For example we would like to know how the spectral property of Paneitz operator affects the topology. Below we will start with dimension 4, when the Q curvature equation and its applications is relatively well understood. Then we will discuss recent progress in dimension $n \geq 5$ about the Green's function of Paneitz operator and the solution to finding constant Q curvature in a fixed conformal class. At last we will turn to the dimension 3, where the Q curvature equation is particularly intriguing and of very different nature from the scalar curvature equation. Open problems will be pointed out along the way.

2. Dimension 4

A basic fact that makes the Q curvature interesting is its appearance in the Chern-Gauss-Bonnet formula. For a closed 4-manifold (M, g) we have

$$\int_M Q d\mu + \frac{1}{4}\int_M |W|^2 d\mu = 8\pi^2 \chi(M). \tag{2.1}$$

Here W is the Weyl tensor. It follows from the pointwise conformal invariance of $|W|^2 d\mu$ and (2.1) that the Q curvature integral is a global conformal invariant which we denote by κ_g i.e.

$$\kappa_g = \int_M Q_g d\mu_g \tag{2.2}$$

and $\kappa_{\widetilde{g}} = \kappa_g$ for any $\widetilde{g} \in [g]$, the conformal class of g. A basic result about this invariant is the following sharp upper bound:

Theorem 2.1 ([Gu2]). *Let (M, g) be a smooth compact four manifold. If $L_g > 0$, then $\kappa_g \leq 16\pi^2$ with equality holds if and only if (M, g) is conformal diffeomorphic to the standard four sphere.*

Theorem 2.1 follows from an identity found in [HY4]. The identity will have a crucial counterpart in other dimensions.

Theorem 2.2 ([HY4]). *Let (M,g) be a 4-dimensional smooth compact Riemannian manifold with $L_g > 0$. For $p \in M$, let $G_{L,p}$ be the Green's function for L_g with pole at p, then we have $\left|Rc_{G^2_{L,p}g}\right|^2_g$ is bounded and*

$$P\left(\log G_{L,p}\right) = 16\pi^2\delta_p - \frac{1}{2}\left|Rc_{G^2_{L,p}g}\right|^2_g - Q \tag{2.3}$$

in distribution sense.

Choosing 1 as test function in (2.3) we see

$$\int_M Q d\mu = 16\pi^2 - \frac{1}{2}\int_M \left|Rc_{G^2_{L,p}g}\right|^2_g d\mu \leq 16\pi^2.$$

If equality holds, then $Rc_{G^2_{L,p}g} = 0$ and by the relative volume comparison theorem we conclude (M,g) must be conformal equivalent to the standard S^4 (see [HY4, section 5]).

To study the Q curvature equation, it is important that the Paneitz operator be nonnegative with only constant functions in its kernel. A quite general condition ensuring such kind of positivity is given by

Theorem 2.3 ([Gu2]). *Let (M,g) be a smooth compact 4-dimensional Riemannian manifold with $L_g > 0$ and $\kappa_g \geq 0$, then the Paneitz operator $P \geq 0$ and the kernel of P consists of constant functions.*

As an application of Theorem 2.1 and 2.3, we have a general existence result for a conformal metric of constant Q curvature. This is analogous to the existence of constant Gauss curvature metrics in dimension two. Let us consider the following functionals

$$I(w) = \int_M |W|^2 w d\mu - \frac{1}{4}\left(\int_M |W|^2 d\mu\right)\log\left(\frac{1}{\mu(M)}\int_M e^{4w} d\mu\right), \tag{2.4}$$

$$\begin{aligned} & II(w) \\ = & \int_M Pw \cdot w d\mu + 2\int_M Qw d\mu - \frac{1}{2}\left(\int_M Q d\mu\right)\log\left(\frac{1}{\mu(M)}\int_M e^{4w} d\mu\right), \end{aligned} \tag{2.5}$$

and

$$III(w) = \int_M J^2_{e^{2w}g} d\mu_{e^{2w}g} - \int_M J^2 d\mu. \tag{2.6}$$

The Euler-Lagrange equation of functional II is given by

$$Pw + Q - \frac{\int_M Q d\mu}{\int_M e^{4w} d\mu} e^{4w} = 0. \tag{2.7}$$

Or in another word,

$$Q_{e^{2w}g} = const. \tag{2.8}$$

On the other hand, the Euler-Lagrange equation for functional III is

$$-\Delta_{e^{2w}g} J_{e^{2w}g} = 0. \tag{2.9}$$

In [ChY] the general functional $F = \gamma_1 I + \gamma_2 II + \gamma_3 III$ was studied.

Theorem 2.4 ([ChY]). *If the functional F satisfies*

$$\gamma_2 > 0, \quad \gamma_3 > 0 \tag{2.10}$$

and

$$\kappa = \frac{\gamma_1}{2}\int_M |W|^2 \, d\mu + \gamma_2 \int_M Q d\mu < 16\pi^2\gamma_2, \tag{2.11}$$

then there exists a minimizer for

$$\inf_{w\in H^2(M)} F(w). \tag{2.12}$$

Any minimizer must be smooth. If w is a minimizer and we write $\widetilde{g} = e^{2w}g$, then

$$\frac{\gamma_1}{2}\left|\widetilde{W}\right|^2_{\widetilde{g}} + \gamma_2\widetilde{Q} - \gamma_3\widetilde{\Delta}\widetilde{J} = \frac{\kappa}{\widetilde{\mu}(M)}. \tag{2.13}$$

Moreover for any $\varphi \in H^2(M)$ with

$$\int_M \varphi d\widetilde{\mu} = 0, \tag{2.14}$$

we have

$$\begin{aligned} & \gamma_2 \int_M \widetilde{P}\varphi\cdot\varphi d\widetilde{\mu} + \gamma_3\int_M \left[\left(\widetilde{\Delta}\varphi + \left|\widetilde{\nabla}\varphi\right|^2_{\widetilde{g}}\right)^2 - 2\widetilde{J}\left|\widetilde{\nabla}\varphi\right|^2_{\widetilde{g}}\right] d\widetilde{\mu} \qquad (2.15)\\ \geq & \frac{\kappa}{2}\log\left(\frac{1}{\widetilde{\mu}(M)}\int_M e^{4\varphi}d\widetilde{\mu}\right). \end{aligned}$$

Here $\int_M \widetilde{P}\varphi\cdot\varphi d\widetilde{\mu}$ is understood in distribution sense.

For the functional II, we have a similar existence result.

Theorem 2.5 ([ChY]). *If*

$$\kappa_g = \int_M Q d\mu < 16\pi^2, \tag{2.16}$$

$P \geq 0$ and the kernel of P consists only of constant functions, then

$$\inf_{w\in H^2(M)} II(w) \tag{2.17}$$

is achieved. Any minimizer must be smooth. If w is a minimizer and we write $\widetilde{g} = e^{2w}g$, then

$$\widetilde{Q} = \frac{\kappa_g}{\widetilde{\mu}(M)}. \tag{2.18}$$

Moreover for any $\varphi \in H^2(M)$ with

$$\int_M \varphi d\widetilde{\mu} = 0, \tag{2.19}$$

we have

$$\int_M \widetilde{P}\varphi\cdot\varphi d\widetilde{\mu} \geq \frac{\kappa_g}{2}\log\left(\frac{1}{\widetilde{\mu}(M)}\int_M e^{4\varphi}d\widetilde{\mu}\right). \tag{2.20}$$

Here $\int_M \widetilde{P}\varphi\cdot\varphi d\widetilde{\mu}$ is understood in distribution sense.

More results on the existence of conformal metrics with constant Q curvature can be found in [DM]. The main ingredient for Theorem 2.4 and 2.5 is the following version of Adams inequality ([A]):

Theorem 2.6 ([BCY, F]). *Let (M,g) be a smooth compact 4-dimensional Riemannian manifold with $P \geq 0$ and kernel of P consists only of constant functions, then for any $w \in H^2(M)$ with*

$$\int_M w d\mu = 0, \tag{2.21}$$

we have

$$\int_M \exp\left(32\pi^2 \frac{w^2}{\int_M Pw \cdot w d\mu}\right) d\mu \leq c(M,g) < \infty. \tag{2.22}$$

In particular

$$\log\left(\frac{1}{\mu(M)} \int_M e^{4w} d\mu\right) \leq \frac{1}{8\pi^2} \int_M Pw \cdot w d\mu + c(M,g). \tag{2.23}$$

Here $\int_M Pw \cdot w d\mu$ is understood in distribution sense.

Adams inequality was discovered in [A] with the motivation of simplifying the original proof in [M]. In particular a higher order sharp inequality was derived through the O'Neil inequality for convolution operator (see [O]) and an one dimensional calculus lemma due to Adams-Garsia. Theorem 2.6 can be proven by modifying O'Neil inequality and the calculus lemma.

For some geometrical and topological applications of these related equations we refer the readers to [ChGY1, ChGY2, Gu1].

3. Dimension at least 5

The analysis of Q curvature and Paneitz operator in dimension greater than 4 has some similarity to the analysis of scalar curvature and conformal Laplacian operator in dimension greater than 2. The research related to Yamabe problem serves as a nice model for asking interesting questions in the study of Paneitz operator. However due to the fact second order differential equations are much better understood than higher order differential equations, sometime the analogous problem for Q curvature can be more challenging.

Based on the fact the first eigenfunction of conformal Laplacian operator can always be chosen as positive everywhere, it was observed in [KW] that in a fixed conformal class, we can always find a metric whose scalar curvature is only of one sign i.e. the scalar curvature is either strictly positive, or identically zero, or strictly negative.

Problem 3.1. *Let (M,g) be a smooth compact Riemannian manifold with dimension $n \geq 5$, can we always find a conformal metric $\widetilde{g}$ such that $\widetilde{Q}$ is either strictly positive, or identically zero, or strictly negative?*

This seems to be a difficult question. One of the obstacle is fourth order symmetric elliptic operators can have no positive first eigenfunction at all. Indeed let M be any smooth compact Riemannian manifold, λ be the smallest positive eigenvalue of $-\Delta$, then the first eigenfunction of $(-\Delta)^2 + 2\lambda\Delta$ must change sign. Though the answer to Problem 3.1 remains mysterious, partial solution to a related problem was found recently in [HY4]. Recall on a smooth compact Riemannian manifold (M,g) with dimension greater than 2, we have

$$\exists \widetilde{g} \in [g] \text{ with } \widetilde{R} > 0 \Longleftrightarrow \lambda_1(L_g) > 0.$$

Here $[g]$ denotes the conformal class of metrics associated with g. The same statement remains true if we replace ">" by "<" or "=" (see [LP]). It is worth pointing out the sign of $\lambda_1(L_g)$ is a conformal invariant. In particular the above statement gives a conformal invariant condition which is equivalent to the existence of a conformal metric with positive scalar curvature.

Problem 3.2. *Let (M, g) be a smooth compact Riemannian manifold with dimension $n \geq 5$, can we find a conformal invariant condition which is equivalent to the existence of a conformal metric with positive Q curvature? Same questions can be asked when "positive" is replaced by "negative" or "zero".*

[HY4] gives a partial answer to this problem under the assumption the Yamabe invariant $Y(g) > 0$.

Theorem 3.1 ([HY4])**.** *Let $n \geq 5$ and (M^n, g) be a smooth compact Riemannian manifold with Yamabe invariant $Y(g) > 0$, then the following statements are equivalent*

(1) $\exists \widetilde{g} \in [g]$ *with* $\widetilde{Q} > 0$.
(2) $\ker P_g = 0$ *and the Green's function of Paneitz operator* $G_P(p, q) > 0$ *for any* $p, q \in M, p \neq q$.
(3) $\ker P_g = 0$ *and there exists a* $p \in M$ *such that* $G_P(p, q) > 0$ *for* $q \in M \backslash \{p\}$.

By transformation law (1.7) we know $\ker P_g = 0$ is a conformal invariant condition, moreover under this assumption, the Green's functions of Paneitz operator G_P satisfy

$$G_{P,\rho^{\frac{4}{n-4}}g}(p, q) = \rho(p)^{-1} \rho(q)^{-1} G_{P,g}(p, q). \tag{3.1}$$

In particular, the fact $G_P > 0$ is also a conformal invariant condition. Of course this condition is clearly more complicated than the one given for the scalar curvature case, however the main strength of Theorem 3.1 lies in that it gives an easy to check necessary and sufficient condition for the positivity of the Green's function of Paneitz operator for metrics of positive Yamabe class. As we will see shortly, the positivity of Green's function is crucial in the study of Q curvature equation.

The main ingredients in proof of Theorem 3.1 is an identity similar to (2.3) in higher dimension.

Theorem 3.2 ([HY4])**.** *Assume $n \geq 5$, (M^n, g) is a smooth compact Riemannian manifold with $Y(g) > 0$, $p \in M$, then we have $G_{L,p}^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}}g} \right|_g^2 \in L^1(M)$ and*

$$P\left(G_{L,p}^{\frac{n-4}{n-2}}\right) = c_n \delta_p - \frac{n-4}{(n-2)^2} G_{L,p}^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}}g} \right|_g^2 \tag{3.2}$$

in distribution sense. Here

$$c_n = 2^{-\frac{n-6}{n-2}} n^{\frac{2}{n-2}} (n-1)^{-\frac{n-4}{n-2}} (n-2)(n-4) \omega_n^{\frac{2}{n-2}}, \tag{3.3}$$

ω_n is the volume of unit ball in $\mathbb{R}^n$, $G_{L,p}$ is the Green's function of conformal Laplacian operator with pole at p.

Here we will give another conformal invariant condition for the existence of conformal metric with positive Q curvature. To achieve this we first introduce some notations.

Let (M,g) be a smooth compact Riemannian manifold. If $K = K(p,q)$ is a suitable function on $M \times M$, we define an operator T_K as

$$T_K(\varphi)(p) = \int_M K(p,q)\,\varphi(q)\,d\mu(q) \tag{3.4}$$

for any nice function φ on M. If $K' = K'(p,q)$ is another function on $M \times M$, then we write

$$(K * K')(p,q) = \int_M K(p,s)\,K'(s,q)\,d\mu(s). \tag{3.5}$$

If $n \geq 5$ and $Y(g) > 0$, we write

$$\begin{aligned} & H(p,q) \qquad (3.6)\\ = \;& 2^{\frac{n-6}{n-2}} n^{-\frac{2}{n-2}} (n-1)^{\frac{n-4}{n-2}} (n-2)^{-1} (n-4)^{-1} \omega_n^{-\frac{2}{n-2}} G_L(p,q)^{\frac{n-4}{n-2}}, \end{aligned}$$

and

$$\begin{aligned} & \Gamma_1(p,q) \qquad (3.7)\\ = \;& 2^{\frac{n-6}{n-2}} n^{-\frac{2}{n-2}} (n-1)^{\frac{n-4}{n-2}} (n-2)^{-3} \omega_n^{-\frac{2}{n-2}} G_L(p,q)^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}} g} \right|_g^2 (q). \end{aligned}$$

Then (3.2) becomes

$$P_q H(p,q) = \delta_p(q) - \Gamma_1(p,q). \tag{3.8}$$

Note that by the calculation in [HY4, Section 2],

$$\Gamma_1(p,q) = O\left(\overline{pq}^{4-n}\right), \tag{3.9}$$

here $\overline{pq}$ denotes the distance between p and q. Assume for all $p \in M$,

$$0 \leq \int_M \Gamma_1(p,q)\,d\mu(q) \leq \alpha < \infty, \tag{3.10}$$

then

$$\|T_{\Gamma_1}\varphi\|_{L^\infty(M)} \leq \alpha \|\varphi\|_{L^\infty(M)}. \tag{3.11}$$

Moreover if we let $\widetilde{g} = \rho^{\frac{4}{n-4}} g$, here ρ is a positive smooth function, then for any smooth function φ on M,

$$T_{\widetilde{\Gamma}_1}(\varphi) = \rho^{-1} T_{\Gamma_1}(\rho\varphi). \tag{3.12}$$

In another word, $T_{\widetilde{\Gamma}_1}$ is similar to T_{Γ_1}. Hence they have the same spectrum and spectral radius i.e. $\sigma\left(T_{\widetilde{\Gamma}_1}\right) = \sigma\left(T_{\Gamma_1}\right)$ and $r_\sigma\left(T_{\widetilde{\Gamma}_1}\right) = r_\sigma\left(T_{\Gamma_1}\right)$ (the spectral radius).

Theorem 3.3. *Assume $n \geq 5$, (M^n, g) is a smooth compact Riemannian manifold with $Y(g) > 0$, then*

$$\exists \widetilde{g} \in [g] \;\; \textit{with } \widetilde{Q} > 0. \Longleftrightarrow \textit{the spectral radius } r_\sigma\left(T_{\Gamma_1}\right) < 1.$$

Moreover if $r_\sigma\left(T_{\Gamma_1}\right) < 1$, then $\ker P = 0$ and

$$G_P = H + \sum_{k=1}^{\infty} \Gamma_k * H, \tag{3.13}$$

here

$$\Gamma_k = \Gamma_1 * \cdots * \Gamma_1 \;\; \textit{(k times)}, \tag{3.14}$$

H and Γ_1 are given in (3.6) and (3.7). The convergence in (3.13) is uniform in the sense that

$$G_P - H - \sum_{k=1}^{l} \Gamma_k * H \longrightarrow 0$$

uniformly on $M \times M$ as $l \to \infty$. In particular, $G_P \geq H$, moreover if $G_P(p,q) = H(p,q)$ for some $p \neq q$, then (M, g) is conformal equivalent to the standard S^n.

Proof. Assume there exists a $\widetilde{g} \in [g]$ with $\widetilde{Q} > 0$, then we hope to show $r_\sigma(T_{\Gamma_1}) < 1$. Because $r_\sigma(T_{\Gamma_1}) = r_\sigma\left(T_{\widetilde{\Gamma}_1}\right)$, replacing g with $\widetilde{g}$ we can assume the background metric satisfies $Q > 0$. By (3.8) we know for any smooth function φ,

$$\varphi = T_H(P\varphi) + T_{\Gamma_1}(\varphi). \tag{3.15}$$

Taking $\varphi = 1$ in (3.15) we get

$$\int_M \Gamma_1(p,q)\, d\mu(q) = 1 - \frac{n-4}{2} \int_M H(p,q)\, Q(q)\, d\mu(q). \tag{3.16}$$

Using the fact $Q > 0$ we know there exists a constant α such that

$$\int_M \Gamma_1(p,q)\, d\mu(q) \leq \alpha < 1$$

for all $p \in M$. It follows that

$$\|T_{\Gamma_1}\|_{\mathcal{L}(L^\infty, L^\infty)} \leq \alpha$$

and hence

$$r_\sigma(T_{\Gamma_1}) \leq \alpha < 1.$$

On the other hand, assume $r_\sigma(T_{\Gamma_1}) < \alpha < 1$, then we can find a constant k_0 such that for $k \geq k_0$,

$$\|T_{\Gamma_k}\|_{\mathcal{L}(L^\infty, L^\infty)} < \alpha^k.$$

It follows that

$$\int_M \Gamma_k(p,q)\, d\mu(q) < \alpha^k.$$

Fix $m > \frac{n}{4}$, using estimate (3.9) we see for all $k \geq k_0 + m$,

$$\|\Gamma_k\|_{L^\infty} \leq \alpha^{k-m} \|\Gamma_m\|_{L^\infty} \leq c\alpha^k.$$

In particular $\|\Gamma_k\|_{L^\infty} \to 0$ and

$$\|\Gamma_k * H\|_{L^\infty} \leq c \|\Gamma_k\|_{L^\infty} \leq c\alpha^k.$$

Iterating (3.15) we see

$$\varphi = T_{H + \Gamma_1 * H + \cdots + \Gamma_{k-1} * H}(P\varphi) + T_{\Gamma_k}(\varphi).$$

Let $k \to \infty$, we see

$$\varphi = T_{H + \sum_{k=1}^{\infty} \Gamma_k * H}(P\varphi).$$

In particular, $P\varphi = 0$ implies $\varphi = 0$ i.e. $\ker P = 0$. Moreover

$$G_P = H + \sum_{k=1}^{\infty} \Gamma_k * H.$$

In particular $G_P \geq H > 0$. If $G_P(p,q) = H(p,q)$ for some $p \neq q$, then $\Gamma_1(p, \cdot) = 0$, in another word

$$Rc_{G_{L,p}^{\frac{4}{n-2}} g} = 0.$$

Since $\left(M\backslash\{p\}, G_{L,p}^{\frac{4}{n-2}}g\right)$ is asymptotically flat, it follows from relative volume comparison theorem that $\left(M\backslash\{p\}, G_{L,p}^{\frac{4}{n-2}}g\right)$ is isometric to $\mathbb{R}^n$, hence (M,g) is conformal equivalent to standard S^n.

Since $G_P > 0$, it follows from Theorem 3.1 that there exists $\widetilde{g} \in [g]$ with $\widetilde{Q} > 0$. □

We remark that the infinite series expansion of G_P in (3.13) is similar to those for Green's function of Laplacian in [Au1].

Remark 3.1. *Indeed it follows from (3.16) that as long as $Y(g) > 0$ and*

$$\int_M H(p,q)\, Q(q)\, d\mu(q) > 0$$

for all $p \in M$, then $r_\sigma(T_{\Gamma_1}) < 1$. In particular this is the case when $Q \geq 0$ and not identically zero.

Problem 3.3. *Let (M,g) be a smooth compact Riemannian manifold with dimension $n \geq 5$, can we find a metric $\widetilde{g} \in [g]$ such that $\widetilde{Q} = const$?*

This turns out to be a difficult problem with only partial solutions available. If we write the unknown metric $\widetilde{g} = \rho^{\frac{4}{n-4}}g$, then we need to solve

$$P\rho = const \cdot \rho^{\frac{n+4}{n-4}}, \quad \rho \in C^\infty(M), \rho > 0. \tag{3.17}$$

As in the case of Yamabe problem, (3.17) has a variational structure. Indeed, for $u \in C^\infty(M)$, let

$$\begin{aligned} E(u) &= \int_M Pu \cdot u d\mu \qquad (3.18)\\ &= \int_M \left[(\Delta u)^2 - 4A(\nabla u, \nabla u) + (n-2)J|\nabla u|^2 + \frac{n-4}{2}Qu^2\right] d\mu. \end{aligned}$$

Clearly we can extend $E(u)$ continuously to $u \in H^2(M)$. Let

$$Y_4(g) = \inf_{u \in H^2(M)\backslash\{0\}} \frac{E(u)}{\|u\|^2_{L^{\frac{2n}{n-4}}}}, \tag{3.19}$$

then $Y_4(g)$ is a conformal invariant in the same spirit as $Y(g)$. If $Y_4(g)$ is achieved at a smooth positive function ρ, then it satisfies (3.17). On the other hand, even if $Y_4(g)$ is achieved at a function $u \in H^2(M)$, we can not conclude whether u changes sign or not. An observation made in [R] says that if $P > 0$ and $G_P > 0$, then the minimizer must be smooth and either strictly positive or strictly negative. We remark that it had been observed in [HeR1, HeR2, HuR] that the positivity of Green's function of Paneitz operator plays crucial roles in various issues related to Q curvature. Without the classical maximum principle, it is hard to know the sign of Green's function of the fourth order operator. A breakthrough was made in [GuM], which provides an easy to check sufficient condition for the positivity of Green's function.

Theorem 3.4 ([GuM])**.** *Assume $n \geq 5$, (M^n, g) is a smooth compact Riemannian manifold with $R > 0$, $Q \geq 0$ and not identically zero, then $P > 0$. Moreover if u is a nonzero smooth function with $Pu \geq 0$, then $u > 0$ and $R_{u^{\frac{4}{n-4}}g} > 0$. In particular, $G_P > 0$.*

Note that the necessary and sufficient condition in Theorem 3.1 is motivated by [GuM, HuR]. The final solution of Yamabe problem uses the positive mass theorem (see [LP, S]). The corresponding statement for the Paneitz operator is established in [GuM, HuR]. Indeed an elementary but ingenious calculation in [HuR] justifies the positivity of mass under the assumption of positivity of Green's function of Paneitz operator for locally conformally flat manifolds. As pointed out in [GuM], the same calculation carries through to nonlocally conformally flat manifolds in dimension $5, 6$ and 7 as well. A close connection between the positive mass result and formula (3.2) is found in [HY4, section 6]. Combine these with Theorem 3.1 and 3.3 we have

Theorem 3.5 ([GuM, HY4, HY5, HuR]). *Assume $n \geq 5$, (M^n, g) is a smooth compact Riemannian manifold with $Y(g) > 0$ and the spectral radius $r_\sigma(T_{\Gamma_1}) < 1$ (Γ_1 is given by (3.7)). If $n = 5, 6, 7$ or (M, g) is locally conformally flat near $p \in M$, then $\ker P = 0$ and under conformal normal coordinate at p, $x_1, \cdots, x_n$,*

$$G_{P,p} = \frac{1}{2n(n-2)(n-4)\omega_n}\left(r^{4-n} + A + O(r)\right),$$

with the constant $A \geq 0$, here $r = |x|$, ω_n is the volume of the unit ball in $\mathbb{R}^n$. Moreover $A = 0$ if and only if (M, g) is conformal equivalent to S^n.

Indeed following [HY4, Section 6] we note that under the assumption of Theorem 3.5 (see [LP])

$$G_{L,p} = \frac{1}{4n(n-1)\omega_n}\left(r^{2-n} + O\left(r^{-1}\right)\right).$$

Let $H_p(q) = H(p, q)$, then

$$G_{P,p} - H_p = \frac{A}{2n(n-2)(n-4)\omega_n} + O(r).$$

It follows from (3.8) that

$$P(G_{P,p} - H_p)(q) = \Gamma_1(p, q).$$

Hence

$$\begin{aligned} & A \\ = \; & 2n(n-2)(n-4)\omega_n \int_M G_P(p, q)\,\Gamma_1(p, q)\,d\mu(q) \\ = \; & 2^{\frac{2(n-4)}{n-2}} n^{\frac{n-4}{n-2}} (n-1)^{\frac{n-4}{n-2}} (n-2)^{-2} (n-4)\omega_n^{\frac{n-4}{n-2}} \int_M G_{P,p} G_{L,p}^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}} g} \right|_g^2 d\mu. \end{aligned}$$

This is exactly the formula proven in [HuR]. Theorem 3.5 follows from this calculation. With Theorem 3.1, 3.3 and 3.5 at hand, we are able to give the first partial solution to Problem 3.3.

Theorem 3.6 ([GuM, HY5]). *Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Y_4(g) > 0$, $r_\sigma(T_{\Gamma_1}) < 1$, then*

1. *$Y_4(g) \leq Y_4(S^n)$, and equality holds if and only if (M, g) is conformally diffeomorphic to the standard sphere.*
2. *$Y_4(g)$ is always achieved. Any minimizer must be smooth and cannot change sign. In particular we can find a constant Q curvature metric in the conformal class.*

(3) *If (M,g) is not conformally diffeomorphic to the standard sphere, then the set of all minimizers u for $Y_4(g)$, after normalizing with $\|u\|_{L^{\frac{2n}{n-4}}}=1$, is compact in C^∞ topology.*

It is worth pointing out that for a locally conformally flat manifold with positive Yamabe invariant and Poincare exponent less than $\frac{n-4}{2}$ (see [SY]), Theorem 3.6 was proved in [QR2] by apriori estimates (using method of moving planes for integral equations developed in [CnLO]) and connecting the equation to Yamabe equation through a path of integral equations.

Note that $Y_4(g)>0$ is the same thing as $P>0$. Either one of the following conditions guarantee the positivity of Paneitz operator

- [GuM, XY1]: $n\geq 5$, $R>0$, $Q\geq 0$ and not identically zero;
- [ChHY, Theorem 1.6]: $n\geq 5$, $J\geq 0$, $\sigma_2(A)\geq 0$ and (M,g) is not Ricci flat.

In applications we are usually interested in metrics not just with $Q>0$, but with both $R>0$ and $Q>0$. This leads us to a question similar to Problem 3.2.

Problem 3.4 ([GuHL, Problem 1.1])**.** *For a smooth compact Riemannian manifold with dimension at least 5, can we find a conformal invariant condition which is equivalent to the existence of a conformal metric with positive scalar and Q curvature?*

Theorem 3.7 ([GuHL])**.** *Let (M,g) be a smooth compact Riemannian manifold with dimension $n\geq 6$. Denote*

$$Y_4^+(g)=\frac{n-4}{2}\inf_{\widetilde{g}\in[g]}\frac{\int_M\widetilde{Q}d\widetilde{\mu}}{(\widetilde{\mu}(M))^{\frac{n-4}{n}}}=\inf_{\substack{u\in C^\infty(M)\\ u>0}}\frac{\int_M Pu\cdot ud\mu}{\|u\|^2_{L^{\frac{2n}{n-4}}}}.$$

and

$$Y_4^*(g)=\frac{n-4}{2}\inf_{\substack{\widetilde{g}\in[g]\\ \widetilde{R}>0}}\frac{\int_M\widetilde{Q}d\widetilde{\mu}}{(\widetilde{\mu}(M))^{\frac{n-4}{n}}}.$$

If $Y(g)>0$ and $Y_4^*(g)>0$, then there exists a metric $\widetilde{g}\in[g]$ satisfying $\widetilde{R}>0$ and $\widetilde{Q}>0$. In particular, $P>0$, the Green's function $G_P>0$, and $Y_4(g)$ is achieved at a positive smooth function u with $R_{u^{\frac{4}{n-4}}g}>0$ and $Q_{u^{\frac{4}{n-4}}g}=const.$ Moreover,

$$Y_4(g)=Y_4^+(g)=Y_4^*(g).$$

Corollary 3.1 ([GuHL])**.** *Let (M,g) be a smooth compact Riemannian manifold with dimension $n\geq 6$. Then the following statements are equivalent*

(1) $Y(g)>0, P>0$.
(2) $Y(g)>0, Y_4^*(g)>0$.
(3) *there exists a metric $\widetilde{g}\in[g]$ satisfying $\widetilde{R}>0$ and $\widetilde{Q}>0$.*

Corollary 3.1 answers Problem 3.4 for dimension at least 6. It also tells us in Theorem 3.6, condition $r_\sigma(T_{\Gamma_1})<1$ is implied by the positivity of $Y(g)$ and $Y_4(g)$ when $n\geq 6$. The case $n=5$ still remains open for Problem 3.4.

Problem 3.5. *Let (M,g) be a smooth compact Riemannian manifold with dimension $n\geq 5$, do we have*

$$Y(g)>0, Q>0\Longrightarrow P>0?$$

The answer is probably negative.

This seems to be a subtle question. Indeed from [GuM, XY1], we know when both R and Q are positive, then P is positive definite. If we have $Y(g) > 0$ and $Q > 0$ instead, then some conformal metrics have positive scalar curvature. However the set of metrics with positive scalar curvature may be disjoint with those with positive Q curvature. Nevertheless Theorem 3.1 tells us $\ker P = 0$ and $G_P > 0$. In [HY5], it is shown this is enough to find a constant Q curvature in the conformal class. Together with Theorem 3.3, we have another partial answer to Problem 3.3.

Theorem 3.8 ([HY5])**.** *Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $r_\sigma(T_{\Gamma_1}) < 1$, then $\ker P = 0$, the Green's function of P is positive and there exists a conformal metric $\widetilde{g}$ with $\widetilde{Q} = 1$.*

Note that if the answer to Problem 3.5 is positive, then Theorem 3.8 would follow from Theorem 3.6. Without knowing the positivity of Paneitz operator, we can not use the minimization problem (3.19) to find the constant Q curvature metrics. A different approach was developed in [HY5]. Under the assumption of Theorem 3.8, it follows from Theorem 3.3 that $\ker P = 0$ and $G_P > 0$. If we denote $f = \rho^{\frac{n+4}{n-4}}$, then equation (3.17) becomes

$$T_{G_P} f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}, \quad f \in C^\infty(M), f > 0. \tag{3.20}$$

Let

$$\Theta_4(g) = \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M T_{G_P} f \cdot f d\mu}{\|f\|^2_{L^{\frac{2n}{n+4}}}}. \tag{3.21}$$

By (3.1), we know $\Theta_4(g)$ is a conformal invariant, moreover it has a nice geometrical description, which is local, (see [HY5, Section 2.1])

$$\begin{aligned} \Theta_4(g) &= \frac{2}{n-4} \sup \left\{ \frac{\int_M \widetilde{Q} d\widetilde{\mu}}{\left\|\widetilde{Q}\right\|^2_{L^{\frac{2n}{n+4}}(M, d\widetilde{\mu})}} : \widetilde{g} \in [g] \right\} \\ &= \sup_{u \in W^{4, \frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M Pu \cdot u d\mu}{\|Pu\|^2_{L^{\frac{2n}{n+4}}}}. \end{aligned} \tag{3.22}$$

It follows from the classical Hardy-Littlewood-Sobolev inequality $\Theta_4(g)$ is always finite. The benefit of this formulation is if $\Theta_4(g)$ is achieved by a maximizer f, we deduce easily from the positivity of G_P that f cannot change sign. With Theorem 3.1, 3.3 and 3.5 at hands, we have the following statement about extremal problem for $\Theta_4(g)$:

Theorem 3.9 ([HY5])**.** *Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $r_\sigma(T_{\Gamma_1}) < 1$, then*

1. *$\Theta_4(g) \geq \Theta_4(S^n)$, here S^n has the standard metric. $\Theta_4(g) = \Theta_4(S^n)$ if and only if (M, g) is conformally diffeomorphic to the standard sphere.*
2. *$\Theta_4(g)$ is always achieved. Any maximizer f must be smooth and cannot change sign. If $f > 0$, then after scaling we have $G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}$ i.e. $Q_{f^{\frac{4}{n+4}} g} = 1$.*

(3) *If (M, g) is not conformally diffeomorphic to the standard sphere, then the set of all maximizers f for $\Theta_4(g)$, after normalizing with $\|f\|_{L^{\frac{2n}{n+4}}} = 1$, is compact in the C^∞ topology.*

The approach in Theorem 3.9 is motivated from the integral equations considered in [HWY1, HWY2]. Integral equation formulation of the Q curvature equation had been used in [QR2].

At last we note that compactness problem for constant Q curvature metrics in a fixed conformal class has been considered in [HeR1, Li, LyX, QR1, WZ].

4. Dimension 3

As we will see soon, the analysis of Q curvature equation in dimension 3 is very different from those in dimension greater than 4. On the other hand, we expect the scalar curvature and Q curvature plays more dominant role for the geometry of the conformal class and the topology of the underlying manifold in dimension 3 than in dimension greater than 4. Because of this, we will list problems in dimension 3 explicitly even though some of them are similar to those in Section 3.

In dimension 3, the Q curvature is given by

$$\begin{aligned} Q &= -\frac{1}{4}\Delta R - 2|Rc|^2 + \frac{23}{32}R^2 \\ &= -\Delta J - 2|A|^2 + \frac{3}{2}J^2 \\ &= -\Delta J + 4\sigma_2(A) - \frac{1}{2}J^2, \end{aligned} \tag{4.1}$$

here

$$J = \frac{R}{4}, \quad A = Rc - Jg. \tag{4.2}$$

The Paneitz operator is given by

$$\begin{aligned} P\varphi &= \Delta^2\varphi + 4\,\mathrm{div}\left[Rc(\nabla\varphi, e_i)\, e_i\right] - \frac{5}{4}\,\mathrm{div}\left(R\nabla\varphi\right) - \frac{1}{2}Q\varphi \\ &= \Delta^2\varphi + 4\,\mathrm{div}\left(A(\nabla\varphi, e_i)\, e_i\right) - \mathrm{div}\left(J\nabla\varphi\right) - \frac{1}{2}Q\varphi. \end{aligned} \tag{4.3}$$

Here e_1, e_2, e_3 is a local orthonormal frame with respect to g. For any smooth positive function ρ,

$$P_{\rho^{-4}g}\varphi = \rho^7 P_g(\rho\varphi). \tag{4.4}$$

Hence

$$Q_{\rho^{-4}g} = -2\rho^7 P_g(\rho). \tag{4.5}$$

Problem 4.1. *Let (M, g) be a 3 dimensional smooth compact Riemannian manifold, can we always find a conformal metric $\widetilde{g}$ such that $\widetilde{Q}$ is either strictly positive, or identically zero, or strictly negative? Can we find a conformal invariant condition which is equivalent to the existence of a conformal metric with positive Q curvature? Same questions can be asked when "positive" is replaced by "negative" or "zero".*

Unfortunately this simple looking question only has partial solution at this stage.

Theorem 4.1 ([HY4])**.** *Let (M, g) be a smooth compact 3 dimensional Riemannian manifold with $Y(g) > 0$, then the following statements are equivalent:*

(1) $\exists \widetilde{g} \in [g]$ *with* $\widetilde{Q} > 0$.
(2) $\ker P_g = 0$ *and the Green's function* $G_P(p,q) < 0$ *for any* $p, q \in M, p \neq q$.
(3) $\ker P_g = 0$ *and there exists a* $p \in M$ *such that* $G_P(p,q) < 0$ *for* $q \in M \backslash \{p\}$.

By transformation law (4.4) we know $\ker P_g = 0$ is a conformal invariant condition. Under this assumption, the Green's functions satisfy

$$G_{P,\rho^{-4}g}(p,q) = \rho(p)^{-1} \rho(q)^{-1} G_{P,g}(p,q). \tag{4.6}$$

Hence the fact $G_P(p,q) < 0$ for $p \neq q$ is a conformal invariant condition. Theorem 4.1 is based on the following identity:

Theorem 4.2 ([HY4])**.** *Let* (M,g) *be a* 3 *dimensional smooth compact Riemannian manifold with* $Y(g) > 0$, $p \in M$, *then we have* $G_{L,p}^{-1} \left| Rc_{G_{L,p}^4 g} \right|_g^2 \in L^1(M)$ *and*

$$P\left(G_{L,p}^{-1}\right) = -256\pi^2 \delta_p + G_{L,p}^{-1} \left| Rc_{G_{L,p}^4 g} \right|_g^2 \tag{4.7}$$

in distribution sense.

If $Y(g) > 0$, we write

$$H(p,q) = -\frac{G_L(p,q)^{-1}}{256\pi^2}, \tag{4.8}$$

and

$$\Gamma_1(p,q) = \frac{G_L(p,q)^{-1}}{256\pi^2} \left| Rc_{G_{L,p}^4 g} \right|_g^2 (q). \tag{4.9}$$

Then (4.7) becomes

$$P_q H(p,q) = \delta_p(q) - \Gamma_1(p,q). \tag{4.10}$$

Note that by the calculation in [HY4, Section 2],

$$\Gamma_1(p,q) = O\left(\overline{pq}^{-1}\right), \tag{4.11}$$

here $\overline{pq}$ denotes the distance between p and q.

If we let $\widetilde{g} = \rho^{-4} g$, here ρ is a positive smooth function, then for any smooth function φ on M,

$$T_{\widetilde{\Gamma}_1}(\varphi) = \rho^{-1} T_{\Gamma_1}(\rho\varphi). \tag{4.12}$$

Hence $T_{\widetilde{\Gamma}_1}$ and T_{Γ_1} have the same spectrum and spectral radius.

Theorem 4.3. *Let* (M,g) *be a* 3 *dimensional smooth compact Riemannian manifold with* $Y(g) > 0$, *then*

$$\exists \widetilde{g} \in [g] \ \text{ with } \widetilde{Q} > 0. \Longleftrightarrow \text{the spectral radius } r_\sigma(T_{\Gamma_1}) < 1.$$

Moreover if $r_\sigma(T_{\Gamma_1}) < 1$, *then* $\ker P = 0$ *and*

$$G_P = H + \sum_{k=1}^{\infty} \Gamma_k * H, \tag{4.13}$$

here

$$\Gamma_k = \Gamma_1 * \cdots * \Gamma_1 \ (k \text{ times}), \tag{4.14}$$

H *and* Γ_1 *are given in (4.8) and (4.9). The convergence in (4.13) is uniform. In particular,* $G_P \leq H$, *moreover if* $G_P(p,q) = H(p,q)$ *for some* p, q, *then* (M,g) *is conformal equivalent to the standard* S^3.

Proof. The argument is basically same as the proof of Theorem 3.3. If there exists a $\widetilde{g} \in [g]$ with $\widetilde{Q} > 0$, by conformal invariance we can assume the background metric has positive Q curvature. By (4.10) for any smooth function φ,

$$\varphi = T_H(P\varphi) + T_{\Gamma_1}(\varphi). \tag{4.15}$$

Taking $\varphi = 1$ in (4.15) we get

$$\int_M \Gamma_1(p,q)\, d\mu(q) = 1 + \frac{1}{2}\int_M H(p,q)\, Q(q)\, d\mu(q). \tag{4.16}$$

Hence for some α

$$\int_M \Gamma_1(p,q)\, d\mu(q) \leq \alpha < 1$$

for all $p \in M$. It follows that

$$\|T_{\Gamma_1}\|_{\mathcal{L}(L^\infty, L^\infty)} \leq \alpha$$

and

$$r_\sigma(T_{\Gamma_1}) \leq \alpha < 1.$$

On the other hand, assume $r_\sigma(T_{\Gamma_1}) < \alpha < 1$, then we can find a constant k_0 such that for $k \geq k_0$,

$$\|T_{\Gamma_k}\|_{\mathcal{L}(L^\infty, L^\infty)} < \alpha^k.$$

It follows that

$$\int_M \Gamma_k(p,q)\, d\mu(q) < \alpha^k.$$

Using (4.11) we see for all $k \geq k_0 + 2$,

$$\|\Gamma_k\|_{L^\infty} \leq \alpha^{k-2} \|\Gamma_2\|_{L^\infty} \leq c\alpha^k.$$

In particular $\|\Gamma_k\|_{L^\infty} \to 0$ and

$$\|\Gamma_k * H\|_{L^\infty} \leq c\, \|\Gamma_k\|_{L^\infty} \leq c\alpha^k.$$

The remaining argument goes exactly the same as in the proof of Theorem 3.3. □

Remark 4.1. *Indeed it follows from (4.16) that as long as $Y(g) > 0$ and*

$$\int_M H(p,q)\, Q(q)\, d\mu(q) < 0$$

for all $p \in M$, then $r_\sigma(T_{\Gamma_1}) < 1$. In particular this is the case when $Q \geq 0$ and not identically zero.

It is worth pointing out that if $\ker P = 0$, then because $\delta_p \in H^{-2}(M)$, we see $G_{P,p} \in H^2(M) \subset C^{\frac{1}{2}}(M)$, in particular the Green's function has a value at the pole, $G_{P,p}(p)$. This pole's value plays exactly the same role as the mass for classical Yamabe problem. If $Y(g) > 0$, $r_\sigma(T_{\Gamma_1}) < 1$ and (M, g) is not conformal diffeomorphic to the standard S^3, it follows from Theorem 4.3 that $G_P(p,q) < 0$ for all $p, q \in M$. On the other hand, on the standard S^3, the Green's function of Paneitz operator touches zero exactly at the pole and is negative away from the pole.

Problem 4.2. *Let (M, g) be a 3 dimensional smooth compact Riemannian manifold, can we find a metric $\widetilde{g} \in [g]$ such that $\widetilde{Q} = const$?*

Theorem 4.4 ([HY3, HY4]). *Let (M,g) be a 3 dimensional smooth compact Riemannian manifold with $Y(g)>0$ and $r_\sigma(T_{\Gamma_1})<1$, then there exists $\widetilde{g}\in[g]$ such that $\widetilde{Q}=1$. Moreover as long as (M,g) is not conformal diffeomorphic to the standard S^3, the set $\left\{\widetilde{g}\in[g]:\widetilde{Q}=1\right\}$ is compact in C^∞ topology.*

Indeed let $\widetilde{g}=u^{-4}g$, then $\widetilde{Q}=1$ becomes

$$Pu=-\frac{1}{2}u^{-7},\quad u\in C^\infty(M),u>0. \tag{4.17}$$

We can assume (M,g) is not conformal diffeomorphic to the standard S^3, then it follows from Theorem 4.3 that $\ker P=0$ and $G_P(p,q)<0$ for all $p,q\in M$. Let $K(p,q)=-G_P(p,q)>0$, then (4.17) becomes

$$u=\frac{1}{2}T_K\left(u^{-7}\right). \tag{4.18}$$

For $0\le t\le 1$, we consider a family of integral equations

$$u=\frac{1}{2}T_{(1-t)+tK}\left(u^{-7}\right). \tag{4.19}$$

Elementary apriori estimate for (4.19) based on the fact K is bounded and strictly positive together with a degree theory argument gives us Theorem 4.4 (see [HY3]). Note the proof of Theorem 4.4 is technically simpler than the proof of Theorem 3.8. This gives a partial solution to Problem 4.2.

To find more solutions to Problem 4.2, we turn our attention to variational methods. If we write $\widetilde{g}=\rho^{-4}g$, then the problem becomes

$$P\rho=const\cdot\rho^{-7},\quad \rho\in C^\infty(M),\rho>0. \tag{4.20}$$

For $u\in C^\infty(M)$, we denote

$$\begin{aligned} E(u,v) &= \int_M Pu\cdot u d\mu \qquad (4.21)\\ &= \int_M\left[(\Delta u)^2-4Rc(\nabla u,\nabla u)+\frac{5}{4}R|\nabla u|^2-\frac{1}{2}Qu^2\right]d\mu\\ &= \int_M\left[(\Delta u)^2-4A(\nabla u,\nabla u)+J|\nabla u|^2-\frac{1}{2}Qu^2\right]d\mu. \end{aligned}$$

It is clear that $E(u)$ extends continuously to $u\in H^2(M)$. Sobolev embedding theorem tells us $H^2(M)\subset C^{\frac{1}{2}}(M)$, hence we can set

$$Y_4(g)=\inf_{u\in H^2(M),u>0}E(u)\left\|u^{-1}\right\|_{L^6}^2=-\frac{1}{2}\sup_{\widetilde{g}\in[g]}\widetilde{\mu}(M)^{\frac{1}{3}}\int_M\widetilde{Q}d\widetilde{\mu}. \tag{4.22}$$

$Y_4(g)$ is a conformal invariant similar to $Y(g)$. But unlike $Y(g)$, it is not clear anymore whether $Y_4(g)$ is finite or not.

Problem 4.3. *Let (M,g) be a 3 dimensional smooth compact Riemannian manifold, do we have $Y_4(g)>-\infty$? Is $Y_4(g)$ always achieved?*

To better understand the problem, following [HY1], we start with some basic analysis. Let u_i be a minimizing sequence for (4.22). By scaling we can assume $\|u_i\|_{L^2}=1$. By Holder inequality we have

$$c=\|1\|_{L^{\frac{3}{2}}}\le\|u_i\|_{L^2}\left\|u_i^{-1}\right\|_{L^6},$$

hence

$$\left\|u_i^{-1}\right\|_{L^6} \geq c > 0.$$

It follows that $E(u_i) \leq c$ and hence $\|u_i\|_{H^2} \leq c$. After passing to a subsequence we can find $u \in H^2(M)$ such that $u_i \rightharpoonup u$ weakly in $H^2(M)$. It follows that $\|u\|_{L^2} = 1$ and $u \geq 0$.

If $u > 0$, then by lower semicontinuity we know u is a minimizer. On the other hand if u touches zero somewhere, then

$$\infty = \left\|u^{-1}\right\|_{L^6} \leq \lim \inf_{i \to \infty} \left\|u_i^{-1}\right\|_{L^6},$$

hence

$$E(u) \leq \lim \inf_{i \to \infty} E(u_i) \leq 0.$$

If we can rule out the second case, then $Y_4(g)$ is achieved.

Definition 4.1 ([HY1]). *Let (M, g) be a 3 dimensional smooth compact Riemannian manifold. If $u \in H^2(M)$ with $u \geq 0$ and $u = 0$ somewhere would imply $E(u) \geq 0$, then we say the metric g (or the associated Paneitz operator) satisfies condition NN^+. If $u \in H^2(M)$ is a nonzero function with $u \geq 0$ and $u = 0$ somewhere would imply $E(u) > 0$, then we say the metric g satisfies condition P^+.*

Theorem 4.5 ([HY1]). *Let (M, g) be a 3 dimensional smooth compact Riemannian manifold. Then we have*

$$Y_4(g) \text{ is finite} \Rightarrow g \text{ satisfies } NN^+$$

and

$$g \text{ satisfies } P^+ \Rightarrow Y_4(g) \text{ is achieved and hence finite.}$$

Note condition P^+ is clearly satisfied when $P > 0$. In this case, Theorem 4.5 was proved in [XY2]. Here is an example when we have positivity of the Paneitz operator.

Lemma 4.1 ([HY1]). *If $Y(g) > 0, \sigma_2(A) > 0$, $Q \leq 0$ and not identically zero, then $P > 0$.*

Examples satisfying assumptions in Lemma 4.1 can be found in Berger spheres (see [HY1]). Here we give another criterion for positivity in the same spirit as [ChHY, Theorem 1.6].

Lemma 4.2. *If $\sigma_2(A) < 0$ and $2Jg \geq A$ (note this implies $J \geq 0$), then $P > 0$.*

Proof. Let

$$\Theta = D^2 u - \frac{\Delta u}{3} g$$

be the traceless Hessian and

$$\mathring{A} = A - \frac{J}{3} g$$

be the traceless Schouten tensor. For convenience we use $A \sim B$ to mean $\int_M A d\mu = \int_M B d\mu$. First we derive the Bochner identity,

$$\begin{aligned} (\Delta u)^2 &= u_{ii} u_{jj} \sim -u_{iij} u_j = -\left(u_{iji} - R_{ijik} u_k\right) u_j \sim u_{ij} u_{ij} + R c_{jk} u_j u_k \\ &= \left|D^2 u\right|^2 + R c(\nabla u, \nabla u) = \left|D^2 u\right|^2 + A(\nabla u, \nabla u) + J |\nabla u|^2. \end{aligned}$$

Hence

$$(\Delta u)^2 \sim |\Theta|^2 + \frac{(\Delta u)^2}{3} + A(\nabla u, \nabla u) + J|\nabla u|^2.$$

In another way

$$(\Delta u)^2 \sim \frac{3}{2}|\Theta|^2 + \frac{3}{2}A(\nabla u, \nabla u) + \frac{3}{2}J|\nabla u|^2.$$

The next step is to remove the ΔJ term in Q curvature. Note that

$$\begin{aligned} -\Delta J \cdot u^2 &= -J_{ii}u^2 \sim 2J_i u \cdot u_i = 2A_{ijj}u \cdot u_i \sim -2A_{ij}u_i u_j - 2A_{ij}u_{ij}u \\ &= -2A_{ij}u_i u_j - 2A_{ij}\Theta_{ij}u - \frac{2}{3}Ju\Delta u \\ &= -2A_{ij}u_i u_j - 2A_{ij}\Theta_{ij}u - \frac{1}{3}J\left(\Delta u^2 - 2|\nabla u|^2\right) \\ &\sim -2A_{ij}u_i u_j - 2A_{ij}\Theta_{ij}u - \frac{1}{3}\Delta J \cdot u^2 + \frac{2}{3}J|\nabla u|^2. \end{aligned}$$

Hence

$$\begin{aligned} -\Delta J \cdot u^2 &\sim -3A(\nabla u, \nabla u) - 3A_{ij}\Theta_{ij}u + J|\nabla u|^2 \\ &= -3A(\nabla u, \nabla u) - 3\mathring{A}_{ij}\Theta_{ij}u + J|\nabla u|^2. \end{aligned}$$

It follows that

$$\begin{aligned} & (\Delta u)^2 + J|\nabla u|^2 - 4A(\nabla u, \nabla u) - \frac{1}{2}Qu^2 \\ \sim\ & \frac{3}{2}|\Theta|^2 + \frac{3}{2}\mathring{A}_{ij}\Theta_{ij}u + 2J|\nabla u|^2 - A(\nabla u, \nabla u) - \frac{3}{4}J^2u^2 + |A|^2u^2 \\ =\ & \frac{3}{2}\left|\Theta + \frac{1}{2}u\mathring{A}\right|^2 + 2J|\nabla u|^2 - A(\nabla u, \nabla u) - \frac{5}{8}\left(J^2 - |A|^2\right)u^2. \end{aligned}$$

In another word

$$\begin{aligned} E(u) &= \frac{3}{2}\int_M \left|\Theta + \frac{1}{2}u\mathring{A}\right|^2 d\mu + \int_M \left[2J|\nabla u|^2 - A(\nabla u, \nabla u)\right] d\mu \\ &\quad -\frac{5}{8}\int_M \left(J^2 - |A|^2\right)u^2 d\mu. \end{aligned}$$

The positivity follows. □

The assumption in Lemma 4.2 is satisfied by $S^2 \times S^1$ with the product metric and some Berger's spheres (see [HY1]).

Conditions P^+ and NN^+ are hard to check in general, on the other hand, they are hard to use too. The closely related conditions P and NN can be introduced.

Definition 4.2 ([HY1]). *Let (M, g) be a 3 dimensional smooth compact Riemannian manifold. If $u \in H^2(M)$ with $u = 0$ somewhere would imply $E(u) \geq 0$, then we say the metric g (or the associated Paneitz operator) satisfies condition NN. If $u \in H^2(M)$ is a nonzero function with $u = 0$ somewhere would imply $E(u) > 0$, then we say the metric g satisfies condition P.*

Condition NN can be used to identify the limit function u, when u touches zero in the brief discussion after Problem 4.3 (see [HY1]).

The standard sphere S^3 does not satisfy condition P^+. Indeed, let x be the coordinate given by the stereographic projection with respect to north pole N, then the Green's function of P at N can be written as

$$G_N = -\frac{1}{4\pi}\frac{1}{\sqrt{|x|^2+1}}. \tag{4.23}$$

In particular, $E(G_N) = G_N(N) = 0$.

Theorem 4.6 ([YgZ]). *$Y_4\left(S^3, g_{S^3}\right)$ is achieved at the standard metric.*

Indeed [YgZ] shows $Y_4\left(S^3\right)$ is achieved by the method of symmetrization. All the critical points are classified by [X]. In [H, HY1], several different approaches are given. The main ingredient is the following observation:

Lemma 4.3 ([HY1]). *Let $N \in S^3$ be the north pole, $u \in H^2\left(S^3\right)$ such that $u(N) = 0$. Denote x as the coordinate given by the stereographic projection with respect to N and*

$$\tau = \sqrt{\frac{|x|^2+1}{2}}.$$

Then we know $\Delta(\tau u) \in L^2\left(\mathbb{R}^3\right)$ and

$$E(u) = \int_{\mathbb{R}^3} |\Delta(\tau u)|^2 \, dx, \tag{4.24}$$

here Δ is the Euclidean Laplacian.

In particular S^3 satisfies NN. The only functions touching 0 and having nonpositive energy are constant multiples of Green's functions.

To help understanding the condition NN, in [HY2], new quantities $\nu(M,g,p)$ and $\nu(M,g)$ are introduced. Let (M,g) be a 3 dimensional smooth compact Riemannian manifold, for any $p \in M$, define

$$\nu(M,g,p) = \inf\left\{\frac{E(u)}{\int_M u^2 d\mu} : u \in H^2(M)\setminus\{0\}, u(p) = 0\right\}. \tag{4.25}$$

When no confusion could arise we denote it as $\nu(g,p)$ or ν_p. We also define

$$\begin{aligned} & \nu(M,g) \\ = & \inf_{p\in M} \nu(M,g,p) \\ = & \inf\left\{\frac{E(u)}{\int_M u^2 d\mu} : u \in H^2(M)\setminus\{0\}, u(p) = 0 \text{ for some } p\right\}. \end{aligned} \tag{4.26}$$

The importance of $\nu(M,g)$ lies in that g satisfies condition P if and only if $\nu(g) > 0$ and it satisfies condition NN if and only if $\nu(g) \geq 0$. It follows from Lemma 4.3 that $\nu\left(S^3, g_{S^3}\right) = 0$. A closely related fact is that the Green's function of Paneitz operator on S^3 vanishes at the pole. In [HY2], first and second variation of $G_P(N,N)$ and $\nu\left(S^3, g, N\right)$ are calculated.

Theorem 4.7 ([HY2]). *Let g be the standard metric on S^3 and h be a smooth symmetric $(0,2)$ tensor. Denote $x = \pi_N$, the stereographic projection with respect to N and*

$$\tau = \sqrt{\frac{|x|^2+1}{2}}.$$

Let G_{g+th} be the Green's function of the Paneitz operator P_{g+th}, then

$$\partial_t|_{t=0} G_{g+th}(N,N) = 0 \tag{4.27}$$

and

$$\begin{aligned} &\partial_t^2|_{t=0} G_{g+th}(N,N) \qquad (4.28)\\ = \; &-\frac{1}{64\pi^2}\int_{\mathbb{R}^3}\left(\sum_{ij}\left(\theta_{ikjk}+\theta_{jkik}-(tr\theta)_{ij}-\Delta\theta_{ij}\right)^2-\frac{3}{2}\left(\theta_{ijij}-\Delta tr\theta\right)^2\right)dx. \end{aligned}$$

Here $\theta = \tau^4 h$ and the derivatives θ_{ikjk} etc are partial derivatives in $\mathbb{R}^3$.

In particular,

$$\partial_t^2|_{t=0} G_{g+th}(N,N) \le 0 \tag{4.29}$$

Moreover, $\partial_t^2|_{t=0} G_{g+th}(N,N) = 0$ if and only if $h = L_X g + f\cdot g$ for some smooth vector fields X and smooth function f on S^3.

For $\nu(g+th,N)$ we have

$$\partial_t|_{t=0}\nu(g+th,N) = 0 \tag{4.30}$$

and

$$\partial_t^2|_{t=0}\nu(g+th,N) = -16\,\partial_t^2|_{t=0} G_{g+th}(N,N). \tag{4.31}$$

In [HY2], a close relation between condition NN and the second eigenvalue of Paneitz operator is given.

Theorem 4.8 ([HY2])**.** *Let (M,g) be a 3 dimensional smooth compact Riemannian manifold with $Y(g) > 0$ and $r_\sigma(T_{\Gamma_1}) < 1$, then the following statements are equivalent:*

1. *$Y_4(g) > -\infty$.*
2. *$\lambda_2(P) > 0$.*
3. *$\nu(g) \ge 0$ i.e. (M,g) satisfies condition NN.*

For condition P, there is a similar statement.

Corollary 4.1 ([HY2])**.** *Let (M,g) be a 3 dimensional smooth compact Riemannian manifold with $Y(g) > 0$ and $r_\sigma(T_{\Gamma_1}) < 1$. If (M,g) is not conformal diffeomorphic to the standard S^3, then the following statements are equivalent:*

1. *$Y_4(g) > -\infty$.*
2. *$\lambda_2(P) > 0$.*
3. *$\nu(g) > 0$ i.e. (M,g) satisfies condition P.*

These statements make the finiteness of $Y_4(g)$ and condition NN more meaningful.

Problem 4.4. *Let (M,g) be a 3 dimensional smooth compact Riemannian manifold, does g always satisfy condition NN? Does metric with positive Yamabe invariant always satisfy condition NN?*

This seems to be a difficult question. We only have a partial answer.

Theorem 4.9 ([HY3, HY4])**.** *Assume M is a smooth compact 3 dimensional manifold, denote*

$$\mathcal{M} = \{g : Y(g) > 0,\, r_\sigma(T_{\Gamma_1}) < 1\}, \tag{4.32}$$

endowed with C^∞ topology. Let $\mathcal{N}$ be a path connected component of $\mathcal{M}$. If there is a metric in $\mathcal{N}$ satisfying condition NN, then every metric in $\mathcal{N}$ satisfies condition NN. Hence as long as the metric is not conformal equivalent to the standard S^3, it satisfies condition P.

Here we describe an application of above discussions. Let M be a 3 dimensional smooth compact manifold, $\beta \in \mathbb{R}$, we define a functional

$$F_\beta(g) = \int_M Q d\mu + \beta \int_M J^2 d\mu. \tag{4.33}$$

Calculation shows that the critical metric of F_β restricted to a fixed conformal class with unit volume constraint is given by

$$Q + 2\beta\Delta J + \beta J^2 = const. \tag{4.34}$$

Proposition 4.1. *Assume (M, g) is a 3 dimensional smooth compact Riemannian manifold satisfying condition P^+, $\beta \leq 0$, then*

$$\sup_{\widetilde{g}\in[g]} \widetilde{\mu}(M)^{\frac{1}{3}} F_\beta(\widetilde{g}) \tag{4.35}$$

is achieved.

Proof. For any positive smooth function u,

$$F_\beta\left(u^{-4}g\right) = -2\int_M Pu \cdot u d\mu + \beta \int_M u^4 \left[-2\Delta\left(u^{-1}\right) + Ju^{-1}\right]^2 d\mu. \tag{4.36}$$

Define

$$\Phi_\beta(u) = \int_M Pu \cdot u d\mu - \frac{\beta}{2}\int_M u^4 \left[-2\Delta\left(u^{-1}\right) + Ju^{-1}\right]^2 d\mu, \tag{4.37}$$

then

$$\sup_{\widetilde{g}\in[g]} \widetilde{\mu}(M)^{\frac{1}{3}} F_\beta(\widetilde{g}) = -2 \inf_{\substack{u\in C^\infty(M)\\ u>0}} \left\|u^{-1}\right\|_{L^6}^2 \Phi_\beta(u). \tag{4.38}$$

Let

$$m = \inf_{\substack{u\in H^2(M)\\ u>0}} \left\|u^{-1}\right\|_{L^6}^2 \Phi_\beta(u). \tag{4.39}$$

We claim m is achieved. Indeed

$$\begin{aligned} \Phi_\beta(u) &= \int_M \left[(\Delta u)^2 - 4A(\nabla u, \nabla u) + J|\nabla u|^2 - \frac{1}{2}Qu^2\right] d\mu \\ &\quad -2\beta\int_M \left(\Delta u - 2u^{-1}|\nabla u|^2 + \frac{J}{2}u\right)^2 d\mu. \end{aligned} \tag{4.40}$$

Assume $u_i \in H^2(M)$, $u_i > 0$ is a minimizing sequence, by scaling we can assume $\max_M u_i = 1$. Then

$$\left\|u_i^{-1}\right\|_{L^6}^2 \Phi_\beta(u_i) \to m.$$

It follows from $\beta \leq 0$ that

$$\left\|u_i^{-1}\right\|_{L^6}^2 E(u_i) \leq c.$$

Hence $E(u_i) \leq c$. Together with the fact $0 < u_i \leq 1$ we get $\|u_i\|_{H^2(M)} \leq c$. After passing to a subsequence we can assume $u_i \rightharpoonup u$ weakly in $H^2(M)$. Then $u_i \to u$ uniformly. It follows that $\max_M u = 1$ and $u \geq 0$. We claim u can not touch 0.

Indeed if u touches zero somewhere, then since $u \in H^2(M)$ we see $\int_M u^{-6} d\mu = \infty$. It follows from Fatou's lemma that

$$\lim_{i\to\infty} \inf \int_M u_i^{-6} d\mu \geq \int_M u^{-6} d\mu = \infty.$$

Hence

$$\lim_{i\to\infty} \sup E(u_i) \leq 0.$$

It follows that $E(u) \leq 0$, this contradicts with condition P^+. The fact $u > 0$ follows. To continue, we observe that $u_i \to u$ in $W^{1,p}(M)$ for $p < 6$. Hence

$$\Phi_\beta(u) \leq \lim_{i\to\infty} \inf \Phi_\beta(u_i).$$

It follows that

$$\left\|u^{-1}\right\|_{L^6}^2 \Phi_\beta(u) \leq \lim_{i\to\infty} \inf \left\|u_i^{-1}\right\|_{L^6}^2 \Phi_\beta(u_i) = m.$$

u is a minimizer. Calculation shows for any $\varphi \in C^\infty(M)$,

$$\begin{aligned} & \int_M uP\varphi d\mu - 2\beta \int_M \left(\Delta u - 2u^{-1}|\nabla u|^2 + \frac{J}{2}u\right) \\ & \cdot \left(\Delta\varphi - 4u^{-1}\langle \nabla u, \nabla\varphi\rangle + 2u^{-2}|\nabla u|^2 \varphi + \frac{J}{2}\varphi\right) d\mu \\ = \quad & const \int_M u^{-7}\varphi d\mu. \end{aligned}$$

Standard bootstrap method shows $u \in C^\infty(M)$. Proposition 4.1 follows. □

Corollary 4.2. *Let (M, g) be a 3 dimensional smooth compact Riemannian manifold satisfying condition P^+, $Y(g) > 0$ and*

$$\int_M Q d\mu - \frac{1}{6}\int_M J^2 d\mu \geq 0, \tag{4.41}$$

then the universal cover of M is diffeomorphic to S^3.

Remark 4.2. *(4.41) is the same as*

$$\int_M |E|^2 d\mu \leq \frac{1}{48}\int_M R^2 d\mu, \tag{4.42}$$

here $E = Rc - \frac{R}{3}g$ is the traceless Ricci tensor.

Proof. It follows from Proposition 4.1 that

$$\kappa = \sup_{\widetilde{g}\in[g]} \widetilde{\mu}(M)^{\frac{1}{3}} \left(\int_M \widetilde{Q} d\widetilde{\mu} - \frac{1}{6}\int_M \widetilde{J}^2 d\widetilde{\mu}\right) \tag{4.43}$$

is achieved. By (4.41) we know $\kappa \geq 0$. Without losing of generality we can assume the background metric g is a maximizer and it has volume 1. Then

$$Q - \frac{1}{3}\Delta J - \frac{1}{6}J^2 = const. \tag{4.44}$$

Integrating both sides we get

$$Q - \frac{1}{3}\Delta J - \frac{1}{6}J^2 = \kappa \geq 0. \tag{4.45}$$

In another way it is

$$-\frac{4}{3}\Delta J - 2\left|\mathring{A}\right|^2 + \frac{2}{3}J^2 = \kappa. \tag{4.46}$$

Here $\mathring{A} = A - \frac{J}{3}g$ is the traceless Schouten tensor. Since the conformal Laplacian is given by $L = -8\Delta + 4J$,

$$LJ = 6\kappa + 12\left|\mathring{A}\right|^2 \geq 0. \tag{4.47}$$

Since $Y(g) > 0$, we see the Green's function of L must be positive, hence either $J > 0$ or $J \equiv 0$. The latter case contradicts with the fact $Y(g) > 0$. So the scalar curvature must be strictly positive. Finally

$$\int_M \sigma_2(A)\, d\mu = \frac{1}{4}\int_M \left(Q + \frac{1}{2}J^2\right) d\mu = \frac{\kappa}{4} + \frac{1}{6}\int_M J^2 d\mu > 0. \tag{4.48}$$

It follows from a result in [CD, GLW] that the universal cover of M is diffeomorphic to S^3. □

The above example also shows the interest in the following question:

Problem 4.5. *Let (M, g) be a 3 dimensional smooth compact Riemannian manifold, can we find a conformal invariant condition which is equivalent to the existence of a conformal metric with positive scalar and Q curvature?*

We remark that by modifying the technique in [GuM], it is shown in [HY3] that on 3 dimensional smooth compact Riemannian manifolds (M, g) with $R > 0$ and $Q > 0$, if $\widetilde{g} \in [g]$ satisfies $\widetilde{Q} > 0$, then $\widetilde{R} > 0$ too.

Acknowledgement. The research of Yang is supported by NSF grant 1104536. We would like to thank Jie Qing and Xingwang Xu for their invitations to talk about the results in this set of lecture notes.

References

[A] D. Adams. *A sharp inequality of J. Moser for higher order derivatives.* Ann. of Math. (2) **128** (1988), no. 2, 385–398.

[Au1] T. Aubin. *Fonction de Green et valeurs propres du laplacien.* J. Math. Pures Appl. (9) **53** (1974), 347–371.

[Au2] T. Aubin. Nonlinear analysis on manifolds, Monge-Ampere equations. Springer Verlag, New York, 1982.

[B] T. Branson. *Differential operators canonically associated to a conformal structure.* Math. Scand. **57** (1985), no. 2, 293–345.

[BCY] T. Branson, S. Y. Chang and P. C. Yang. *Estimates and extremals for zeta function determinants on four-manifolds.* Comm. Math. Phys. **149** (1992), no. 2, 241–262.

[CD] G. Catino and Z. Djadli. *Conformal deformations of integral pinched 3-manifolds.* Adv. Math. **223** (2010), no. 2, 393–404.

[ChGY1] S. Y. Chang, M. J. Gursky and P. C. Yang. *An equation of Monge-Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature.* Ann. of Math. (2) **155** (2002), no. 3, 709–787.

[ChGY2] S. Y. Chang, M. J. Gursky and P. C. Yang. *A conformally invariant sphere theorem in four dimensions.* Publ. Math. Inst. Hautes Etudes Sci. No. **98** (2003), 105–143.

[ChHY] S. Y. Chang, F. B. Hang and P. C.Yang. *On a class of locally conformally flat manifolds.* Int. Math. Res. Not. **2004**, no. 4, 185–209.

[ChY] S. Y. Chang and P. C. Yang. *Extremal metrics of zeta function determinants on 4-manifolds.* Ann. of Math. (2) **142** (1995), no. 1, 171–212.

[CnLO] W. X. Chen, C. M. Li and B. Ou. *Classification of solutions for an integral equation.* Comm Pure Appl Math **59** (2006), no. 3, 330–343.

[DM] Z. Djadli and A. Malchiodi. *Existence of conformal metrics with constant Q curvature.* Ann. of Math. (2) **168** (2008), no. 3, 813–858.

[F] L. Fontana. *Sharp borderline Sobolev inequalities on compact Riemannian manifolds.* Comment. Math. Helv. **68** (1993), no. 3, 415–454.

[GLW] Y. X. Ge, C. S. Lin and G. F. Wang. *On the σ_2-scalar curvature.* J. Differential Geom. **84** (2010), no. 1, 45–86.

[Gu1] M. J. Gursky. *The Weyl functional, de Rham cohomology, and Kahler-Einstein metrics.* Ann. of Math. (2) **148** (1998), no. 1, 315–337.

[Gu2] M. J. Gursky. *The principal eigenvalue of a conformally invariant differential operator.* Comm. Math. Phys. **207** (1999), no. 1, 131–143.

[GuHL] M. J. Gursky, F. B. Hang and Y. J. Lin. *Riemannian manifolds with positive Yamabe invariant and Paneitz operator.* Int. Math. Res. Not. IMRN **2016**, no. 5, 1348–1367.

[GuM] M. J. Gursky and A. Malchiodi. *A strong maximum principle for the Paneitz operator and a nonlocal flow for the Q curvature.* J. Eur. Math. Soc. **17** (2015), 2137–2173.

[H] F. B. Hang. *On the higher order conformal covariant operators on the sphere.* Commun Contemp Math. **9** (2007), no. 3, 279–299.

[HWY1] F. B. Hang, X. D. Wang and X. D. Yan. *Sharp integral inequalities for harmonic functions.* Comm. Pure Appl. Math. **61** (2008), no. 1, 54–95.

[HWY2] F. B. Hang, X. D. Wang and X. D. Yan. *An integral equation in conformal geometry.* Ann. Inst. H. Poincare Anal. Non Lineaire **26** (2009), no. 1, 1–21.

[HY1] F. B. Hang and P. C. Yang. *The Sobolev inequality for Paneitz operator on three manifolds.* Calculus of Variations and PDE. **21** (2004), 57–83.

[HY2] F. B. Hang and P. C. Yang. Paneitz operator for metrics near S^3. Preprint (2014), arXiv:1504.02032.

[HY3] F. B. Hang and P. C. Yang. *Q curvature on a class of* 3 *manifolds.* Comm. Pure Appl. Math. **69** (2016), no. 4, 734–744.

[HY4] F. B. Hang and P. C. Yang. *Sign of Green's function of Paneitz operators and the Q curvature.* Int. Math. Res. Not. IMRN **2015**, no. 19, 9775–9791.

[HY5] F. B. Hang and P. C. Yang. Q curvature on a class of manifolds with dimension at least 5. Comm Pure Appl Math, to appear, doi: 10.1002/cpa.21623.

[HeR1] E. Hebey and F. Robert. *Compactness and global estimates for the geometric Paneitz equation in high dimensions.* Electron Res Ann Amer Math Soc. **10** (2004), 135–141.

[HeR2] E. Hebey and F. Robert. *Asymptotic analysis for fourth order Paneitz equations with critical growth.* Adv Calc Var. **4** (2011), no. 3, 229–275.

[HuR] E. Humbert and S. Raulot. *Positive mass theorem for the Paneitz-Branson operator.* Calculus of Variations and PDE. **36** (2009), 525–531.

[KW] J. L. Kazdan and F. W. Warner. *Scalar curvature and conformal deformation of Riemannian structure.* J. Diff. Geom. **10** (1975), 113–134.

[LP] J. M. Lee and T. H. Parker. *The Yamabe problem.* Bull AMS. **17** (1987), no. 1, 37–91.

[Li] G. Li. A compactness theorem on Branson's Q-curvature equation. Preprint (2015), arXiv:1505.07692.

[LyX] Y. Y. Li and J. G. Xiong. Compactness of conformal metrics with constant Q-curvature. I. Preprint (2015), arXiv:1506.00739.

[M] J. Moser. *A sharp form of an inequality by N. Trudinger.* Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

[O] R. O'Neil. *Convolution operators and L(p,q) spaces.* Duke Math. J. **30** (1963), 129–142.

[P] S. Paneitz. A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds. Preprint (1983), arXiv:0803.4331.

[QR1] J. Qing and D. Raske. *Compactness for conformal metrics with constant Q curvature on locally conformally flat manifolds.* Calc. Var. Partial Differential Equations **26** (2006), no. 3, 343–356.

[QR2] J. Qing and D. Raske. *On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds.* Int Math Res Not. Art. id 94172 (2006).

[R] F. Robert. Fourth order equations with critical growth in Riemannian geometry. Unpublished notes. Available at http://www.iecn.u-nancy.fr/~frobert/LectRobertFourth.pdf.

[S] R. Schoen. *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Differential Geom. **20** (1984), no. 2, 479–495.

[SY] R. Schoen and S. T. Yau. *Conformally flat manifolds, Kleinian groups and scalar curvature.* Inventiones Mathematicae **92** (1988) no. 1, 47–71.

[T] N. Trudinger. *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds.* Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.

[WZ] J. C. Wei and C. Y. Zhao. *Non-compactness of the prescribed Q-curvature problem in large dimensions.* Calc. Var. Partial Differential Equations **46** (2013), no. 1-2, 123–164.

[X] X. W. Xu. *Exact solutions of nonlinear conformally invariant integral equations in* $\mathbb{R}^3$. Adv. Math. **194** (2005), no. 2, 485–503.

[XY1] X. W. Xu and P. C. Yang. *Positivity of Paneitz operators.* Discrete Contin. Dynam. Systems **7** (2001), no. 2, 329–342.

[XY2] X. W. Xu and P. C. Yang. *On a fourth order equation in 3-D.* ESAIM: Control, optimization and Calculus of Variations **8** (2002) 1029–1042.

[Y] H. Yamabe. *On a deformation of Riemannian structures on compact manifolds.* Osaka Math. J. **12** (1960), 21–37.

[YgZ] P. C. Yang and M. J. Zhu. *Sharp inequality for Paneitz operator on* S^3. ESAIM: Control, Optim. and Calc. of Var. **10** (2004) 211–223.

Courant Institute, New York University, 251 Mercer Street, New York NY 10012
E-mail address: fengbo@cims.nyu.edu

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544
E-mail address: yang@math.princeton.edu