We study two resonant Hamiltonian systems on the phase space $L^2(\mathbb{R} \to \mathbb{C})$: the quintic one-dimensional continuous resonant equation, and a cubic resonant system that appears as the modified scattering limit of a specific NLS equation. We prove that these systems approximate the dynamics of the quintic and cubic one-dimensional NLS with harmonic trapping in the small data regime on long times scales. We then pursue a thorough study of the dynamics of the resonant systems themselves. Our central finding is that these resonant equations fit into a larger class of Hamiltonian systems that have many striking dynamical features: non-trivial symmetries such as invariance under the Fourier transform and the flow of the linear Schrödinger equation, a robust wellposedness theory, including global wellposedness in $L^2$, and an infinite family of orthogonal, explicit stationary wave solutions in the form of the Hermite functions.

1. Introduction

In recent years resonant systems have emerged as extremely useful tools for studying nonlinear Schrödinger equations (NLS). Resonant equations have been used to construct solutions of the cubic NLS on $\mathbb{T}^2$ that exhibit large growth of Sobolev norms [10]. They have appeared as modified scattering limits for a number of equations, including the cubic NLS on $\mathbb{R} \times \mathbb{T}^d$ [20], the cubic NLS on $\mathbb{R}^d$ with $2 \leq d \leq 5$ and harmonic trapping in all but one direction [21], and a coupled cubic NLS system on $\mathbb{R} \times \mathbb{T}$ [25]. The continuous resonant equation (CR) was originally shown to approximate the dynamics of small solutions of the two-dimensional cubic NLS on a large torus $\mathbb{T}^2_L$ over long times scales (longer than $L^2/\epsilon^2$, where $\epsilon$ is the size of the initial data) [12]. Recent work has extended this by showing that a whole family of CR equations approximate the dynamics of NLS on $\mathbb{T}^d_L$ for arbitrary dimension and arbitrary analytic nonlinearity [8]. The original two-dimensional cubic CR equation is the same resonant system that appears in the modified scattering limit in [21] for $d = 3$; it has also been shown to be a small data approximation for the cubic NLS with harmonic trapping set on $\mathbb{R}^2$ [17].

One of the principal reasons that resonant systems are useful is that they generally exhibit a large amount of structure. They are often Hamiltonian and usually possess many symmetries, a good wellposedness theory, and an infinite number of orthogonal, explicit solutions. Extensive work has been done on studying such purely dynamical properties of the CR equations: starting in the paper that introduced the original two-dimensional cubic equation [12], in subsequent works again on this cubic case [16, 17], and a more recent paper on the general case [7]. This research fits into a larger program of studying the dynamics of nonlocal Hamiltonian PDEs; we mention, for example, work on the Szegő equation [15] and the lowest Landau level equation [14].

The two-dimensional cubic CR equation has, in particular, been found to have many remarkable dynamical properties. The PDE is symmetric under many non-trivial actions such as the Fourier transform and the linear flow of the Schrödinger equation (with or without harmonic trapping); it is Hamiltonian, and through these symmetries admits a number of conserved quantities. The equation is globally wellposed in $L^2$ and all higher Sobolev spaces. It has many explicit stationary wave solutions, including all of the Hermite functions and the function $1/|x|$. All stationary waves that are in $L^2$ are automatically analytic and exponentially decaying in physical space and Fourier space.

The present work was initiated by the question of whether these striking properties also hold for the only other continuous resonant equation that scales like $L^2$: the one-dimensional quintic continuous

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resonant equation. Our investigation subsequently broadened to include another one-dimensional resonant equation that is somewhat more physically relevant, and turns out to be the modified scattering limit in [21] for $d = 2$. Our overall finding is that these Hamiltonian systems do display much of the remarkable dynamical structure of the two-dimensional cubic CR. In fact, we are able to show that both systems belong to a large class of Hamiltonian systems on the phase space $L^2(\mathbb{R} \to \mathbb{C})$, and that each system is this class bears many of the features of $L^2$ critical CR: they have a strong symmetry structure, global wellposedness in $L^2$ and other Sobolev spaces, and many explicit stationary wave solutions in the form of the Hermite functions. Typical members of the class lack much of the structure of both cubic two-dimensional and quintic one-dimensional CR – for example, it is not the case that all $L^2$ stationary waves are analytic – but our findings do suggest that a number of the properties of the $L^2$ critical CR equations are generic.

1.1. Presentation of the equations. The two systems we study in this article are resonant systems corresponding to the nonlinear Schrödinger equation with harmonic trapping,

\begin{equation}
(1.1) \quad iv_t - \Delta v + v^2 v = iv_x + Hv = |u|^{2k} u,
\end{equation}

where the spatial variable is $x \in \mathbb{R}$ and $k$ is an integer, so that the nonlinearity is analytic. The cubic $k = 1$ equation is physically relevant: in this case, (1.1) is the Gross–Pitaevskii equation and is a model in the physical theory of Bose–Einstein condensates [19].

Let us first see how the resonant equations arise. Looking at the profile $v(t) = e^{-itH} u(t)$ (where $e^{itH}$ is the propagator of the linear equation $iv_t + Hv = 0$), we find it satisfies $iv_t = e^{-itH} \left( |e^{itH} u|^{2k} e^{itH} v \right)$. Expressing $v(t)$ in the basis of eigenfunctions of the operator $H$ (namely the Hermite functions), the equation on $v$ can be written as,

\begin{equation}
(1.2) \quad iv_t(t) = \sum_{n_1, \ldots, n_{2k+2} \in \mathbb{Z}^+} e^{2itL} \Pi_{n_{2k+2}} \left[ \prod_{m=1}^{k} \left( (\Pi_{n_m} v(t))(\Pi_{n_{k+1+m}} v(t)) \right) \Pi_{n_{k+1}} v(t) \right],
\end{equation}

where $\Pi_n v$ is the projection onto the eigenspace of $H$ corresponding to eigenvalue $2n + 1$. The phase $L$ in (1.2) is given by $L = n_1 + \ldots + n_{k+1} - (n_{k+2} + \ldots + n_{2k+2})$. The resonant terms in the sum in (1.2) are the terms that are not oscillating in time; that is, those satisfying $L = 0$. The resonant system corresponding to (1.2) is obtained by considering only the resonant terms; namely,

\begin{equation}
(1.3) \quad iv_t(t) = \sum_{n_1, \ldots, n_{2k+2} \in \mathbb{Z}^+ \atop L=0} \Pi_{n_{2k+2}} \left[ \prod_{m=1}^{k} \left( (\Pi_{n_m} w(t))(\Pi_{n_{k+1+m}} w(t)) \right) \Pi_{n_{k+1}} v(t) \right].
\end{equation}

We will show in Section 2 that this resonant PDE may be written more compactly in terms of a certain time average of the nonlinearity,

\begin{equation}
(1.4) \quad iv_t(t) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} e^{-isH} \left( |e^{isH} w(t)|^{2k} e^{isH} w(t) \right) ds.
\end{equation}

From this expression we are able to infer that the resonant system is the Hamiltonian flow on the phase space $L^2(\mathbb{R} \to \mathbb{C})$ corresponding to the Hamiltonian,

\begin{equation}
(1.5) \quad \mathcal{H}_{2k+2}(f) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} |(e^{itH} f)(x)|^{2k+2} dx dt.
\end{equation}

The overall resonant program is to gain information on the dynamics of solutions to (1.1) by studying the associated resonant system (1.4). This program has two, distinct components. The first is to establish approximation results that rigorously demonstrate that solutions of the resonant system well approximate solutions of the full system in certain function spaces and over certain timescales. The second component of the program is to understand the dynamics of the resonant equation itself. One then projects these dynamics back to the original equation through the approximation results.

We start the article by proving an approximation result that is valid for all positive integers $k$. We then analyze the resonant system (1.4) in depth for the cubic case, when $k = 1$, and the quintic case, when $k = 2$. These two cases are particularly significant for separate reasons.
• The cubic case $k = 1$ is physically relevant, as previously mentioned. In addition, the resonant equation here is exactly the resonant equation obtained in [21] as the modified scattering limit of the NLS equation,

\[ iu_t - u_{xx} - u_{yy} + |y|^2u = |u|^2u, \]

where the space variable is $(x, y) \in \mathbb{R}^2$. Precisely, consider small initial data $u_0(x, y)$. Suppose that $u(x, y, t)$ solves (1.6) with initial data $(x, y) \mapsto u_0(x, y)$. For each fixed $x$, let $w(x, y, t)$ be the solution of the resonant equation (1.3) with initial data $y \mapsto u_0(x, y)$. Then,

\[
\lim_{t \to \infty} \left\| u(x, y, t) - e^{it(-\partial_{xx} - \partial_{yy} + |y|^2)}w(x, y, 2 \ln(t)) \right\|_{H^N(\mathbb{R}^2)} = 0,
\]

where $H^N$ is the usual Sobolev space. (This holds for any $N \geq 8$ so long as the initial data is sufficiently small.)

• In the quintic case, $k = 2$, we will prove that the resonant system (1.4) is precisely the one-dimensional quintic continuous resonant equation. It is the only CR equation, other than the original two-dimensional cubic CR equation, that scales like $L^2$. One of the central motivations of this work is to understand the dynamics of the CR system in this important special case.

1.2. Obtained results.

1.2.1. An approximation theorem. We begin, in Section 2, by proving the following theorem, which shows that solutions of the resonant equation (1.4) well-approximate solutions of the full equation (1.1) on a long time scale. This theorem is essentially a lower dimensional version of Theorem 3.1 in [17], and our proof follows theirs closely.

**Theorem** (Theorem 2.3, page 10). Define the space $\mathcal{H}^s$ by the norm $\|f\|_{\mathcal{H}^s} = \|H^{s/2}f\|_{L^2}$; this is equivalent to the norm $\|\langle x \rangle^s f \|_{L^2} + \|\langle \xi \rangle^s \hat{f} \|_{L^2}$. Fix $s > 1/2$ and initial data $u_0 \in \mathcal{H}^s$. Let $u$ be a solution of the nonlinear Schrödinger equation with harmonic trapping (1.1) and $w$ a solution of the resonant equation (1.4), both corresponding to the initial data $u_0$. Suppose that the bounds $\|u(t)\|_{\mathcal{H}^s}, \|w(t)\|_{\mathcal{H}^s} \leq \epsilon$ hold for all $t \in [0, T]$. Then for all $t \in [0, T]$,

\[
\|u(t) - e^{itH}w(t)\|_{\mathcal{H}^s} \leq (2k + 1)\epsilon^{k+1} + \epsilon^{2k+1} \exp \left( (2k + 1)t\epsilon^{2k} \right).
\]

In particular if $t \leq \epsilon^{-2k}$ then $\|u(t) - e^{itH}w(t)\|_{\mathcal{H}^s} \leq \epsilon^{2k+1}$.

1.2.2. Representation formulas for the Hamiltonians. Following the approximation result, we focus on studying the resonant system (1.4) in the cases $k = 2$ and $k = 1$. Right away we note that the Hamiltonians (1.5) for these systems arise from multilinear functionals, in the following way. In the quintic case ($k = 2$), the Hamiltonian is,

\[
\mathcal{H}_6(f) = \frac{2}{\pi} \|e^{itH}f\|_{L^6_xL^6_t}^6 = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_\mathbb{R} |\langle e^{itH} f \rangle(x)|^6 \, dx \, dt,
\]

which arises from the multilinear functional,

\[ E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_\mathbb{R} \langle e^{itH} f_1 \rangle\langle e^{itH} f_2 \rangle\langle e^{itH} f_3 \rangle\langle e^{itH} f_4 \rangle\langle e^{itH} f_5 \rangle\langle e^{itH} f_6 \rangle \, dx \, dt,
\]

through $\mathcal{H}_6(f) = E_6(f, f, f, f, f, f)$. The cubic Hamiltonian, corresponding to $k = 1$, is,

\[
\mathcal{H}_4(f) = \frac{2}{\pi} \|e^{itH}f\|_{L^4_xL^4_t}^4 = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_\mathbb{R} |\langle e^{itH} f \rangle(x)|^4 \, dx \, dt,
\]

which is associated to the multilinear functional,

\[ E_4(f_1, f_2, f_3, f_4) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_\mathbb{R} \langle e^{itH} f_1 \rangle\langle e^{itH} f_2 \rangle\langle e^{itH} f_3 \rangle\langle e^{itH} f_4 \rangle \, dx \, dt,
\]

through $\mathcal{H}_4(f) = E_4(f, f, f, f)$. The fact that the Hamiltonians can be expressed in terms of multilinear functionals is a nontrivial structural property that guides much of the analysis. The symmetries of the Hamiltonian, its wellposedness theory and the existence of certain stationary wave solutions can
all be determined from studying the associated multilinear functional (see Theorems 3.8 and 3.10 for examples of this in practice). We will prove that Hamilton’s equations corresponding to $H_6$ and $H_4$ are given by,

$$i u_t = T_6(u, u, u, u, u) \quad \text{and} \quad i u_t = T_4(u, u, u),$$

respectively, where the multilinear operators $T_6$ and $T_4$ are defined implicitly by the formulas,

$$\langle T_6(f_1, f_2, f_3, f_4, f_5, g) \rangle_{L^2} = 6 \mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, g) \quad \text{and} \quad \langle T_4(f_1, f_2, f_3, g) \rangle_{L^2} = 4 \mathcal{E}_4(f_1, f_2, f_3, g).$$

Hamilton’s equations are precisely the resonant equations (1.4) in the cases $k = 2$ and $k = 1$.

In the study of resonant equations, it has turned out to be fundamental to determine alternative representations for the Hamiltonian $H$, the associated multilinear functional $\mathcal{E}$ and associated multilinear operator $T$. These alternative representations often reveal structure that is concealed by specific representations such as (1.7) and (1.9). In Sections 4.1 and 5.1 we derive numerous representations for $\mathcal{E}_6$ and $\mathcal{E}_4$ respectively. First, for $\mathcal{E}_6$, we find the two formulas,

\begin{equation}
\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( e^{it \Delta} f_1 \right) \left( e^{it \Delta} f_2 \right) \left( e^{it \Delta} f_3 \right) \left( e^{it \Delta} f_4 \right) \left( e^{it \Delta} f_5 \right) dx dt
\end{equation}

\begin{equation}
= \frac{1}{\pi^2} \int_{\mathbb{R}^6} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) f_5(y_5) f_6(y_6)
\end{equation}

\begin{equation}
\delta_{y_1 + y_2 + y_3 = y_4 + y_5 + y_6} \delta_{y_1^2 + y_2^2 + y_3^2 = y_4^2 + y_5^2 + y_6^2} d y,
\end{equation}

where, in the first equation, $e^{it \Delta}$ denotes the propagator of the linear Schrödinger equation. These representations both show that the quintic Hamiltonian system is the one-dimensional quintic continuous resonant equation [8].

To describe our next representations, we require some notation. For an isometry $A : \mathbb{R}^3 \to \mathbb{R}^3$, let $E_A$ be the multilinear functional,

\begin{equation}
E_A(f_1, f_2, f_3, f_4, f_5, f_6) = \int_{\mathbb{R}^3} f_1((Ax)_1) f_2((Ax)_2) f_3((Ax)_3) f_4(x_1) f_5(x_2) f_6(x_3) dx_1 dx_2 dx_3,
\end{equation}

where $(Ax)_k = \langle Ax, e_k \rangle$. The functional $E_A$ is a special case of the type of functional that appears in Brascamp–Lieb inequalities [6]. We then have the following representations: for the quintic equation, we prove that,

\begin{equation}
\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2 \sqrt{3} \pi^2} \int_0^{2\pi} E_R(\theta)(f_1, f_2, f_3, f_4, f_5, f_6) d \theta,
\end{equation}

where $R(\theta)$ is the rotation of $\mathbb{R}^3$ by $\theta$ radians about the axis $(1, 1, 1)$; while for the cubic equation, we prove that,

\begin{equation}
\mathcal{E}_4(f_1, f_2, f_3, f_4) = \frac{1}{2 \sqrt{2} \pi^2} \int_0^{2\pi} E_S(\theta)(G, f_1, f_2, G, f_3, f_4) d \theta,
\end{equation}

where $G(x) = e^{-x^2/2}$, and $S(\theta)$ is the rotation of $\mathbb{R}^3$ by $\theta$ radians about the axis $(0, 1, 1)$.

The two representations (1.13) and (1.14) are extremely beneficial for studying $H_6$ and $H_4$. They also place the two Hamiltonians in a larger class of Hamiltonians that, we will find, share much of the same structure. This is not obvious: a priori we might expect the Hamiltonians $H_6$ and $H_4$ to be quite unlike. The differences in (1.13) and (1.14) are also of note. The presence of the Gaussians $G$ in (1.14) ultimately causes the symmetry group of the cubic equation to be smaller than that of the quintic equation; it is also prevents the cubic equation from having a scaling law, which has consequences for the possible stationary waves we can construct.

1.2.3. Properties of a class of multilinear functionals. Formulas (1.13) and (1.14) suggest that one can learn much about the dynamics of the Hamiltonian systems $H_6$ and $H_4$ by studying the class of functionals $E_A$. This is precisely what we do in Section 3. Our overall finding is that many of the remarkable properties of the two-dimensional cubic CR equation may be found at the level of the functionals $E_A$. By (1.13) and (1.14), these properties are inherited directly by $\mathcal{E}_6$ and $\mathcal{E}_4$. We will show that the functionals $E_A$ have a large group of symmetries, that the associated PDE problem is locally wellposed in every Sobolev space and globally wellposed in $L^2$, and that all of the Hermite
functions are stationary wave solutions of the associated PDE problem. We emphasize that $A$ is always assumed to be an isometry of $\mathbb{R}^3$, as it is in (1.13) and (1.14).

**Theorem** (Theorems 3.2 and 3.3, page 14). The functional $E_A$ is invariant under the following actions (for any $\lambda$):

(i) Fourier Transform: $f_k \mapsto \hat{f}_k$.

(ii) Modulation: $f_k \mapsto e^{i\lambda} f_k$.

(iii) $L^2$ scaling: $f_k(x) \mapsto \lambda^{1/2} f_k(\lambda x)$.

(iv) Quadratic modulation: $f_k \mapsto e^{i\lambda^2} f_k$.

(v) Schrödinger group: $f_k \mapsto e^{i\lambda H} f_k$.

(vi) Schrödinger with harmonic trapping group: $f_k \mapsto e^{i\lambda H} f_k$.

(vii) Linear modulation: $f_k \mapsto e^{i\lambda x} f_k$.

(viii) Translation: $f_k \mapsto f_k(\cdot + \lambda)$.

All of these actions give symmetries for $\mathcal{H}_6$, by (1.13). For $\mathcal{H}_4$, the symmetries are not inherited automatically because of the Gaussian terms in (1.14), however we find that five of the eight symmetries do hold.

**Corollary** (Theorems 4.8, page 28, and 5.5, page 42). The functional $E_6$ is invariant under all the actions (i) through (viii). The functional $E_4$ is invariant under the actions (i), (ii), (vi), (vii) and (viii).

These symmetries, seen purely at the level of $E_A$, are used to generate conserved quantities for the resonant equations, using Noether’s Theorem.

**Corollary** (Tables 4.2, page 28, and 5.2, page 43). The following are conserved quantities of the resonant equation (1.4) in the cases $k = 2$ and $k = 1$,

$$
\int_\mathbb{R} |f(x)|^2 dx, \quad \int_\mathbb{R} |f(x)|^2 dx, \quad \int_\mathbb{R} i f'(x) \overline{\nabla f(x)} dx, \quad \int_\mathbb{R} |f(x)|^2 + |f'(x)|^2 dx.
$$

In the quintic case $k = 2$, we have the additional conserved quantities,

$$
\int_\mathbb{R} |x f'(x) + f(x)| \overline{\nabla f(x)} dx, \quad \int_\mathbb{R} |f(x)|^2 dx, \quad \int_\mathbb{R} |f'(x)|^2 dx.
$$

We next examine the $L^2$ boundedness of $E_A$.

**Theorem** (Theorem 3.6, page 18). There holds the bound $|E_A(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \prod_{k=1}^{6} \|f_k\|_{L^2}$.

In particular, $|H_4(f)| := |E_A(f, f, f, f, f, f)| \leq \|f\|_{L^2}^6$. If $A$ is not a signed permutation, there is equality in the Hamiltonian bound if and only if $f$ is a Gaussian.

This bound is actually an example of a geometric Brasscamp-Lieb inequality, and the classification of the maximizers is already known [2]. We prove the inequality and classify the maximizers in our case in a way that appears to be new.

The $L^2$ bound on $E_A$ directly gives $L^2$ bounds for $\mathcal{H}_6$ and $\mathcal{H}_4$; indeed, $|\mathcal{H}_6(f)| \leq \frac{1}{2\sqrt{\pi}} \int_0^{2\pi} |E_{R(\theta)}(f, f, f, f, f, f)| d\theta \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^2}^6$, and similarly, $|\mathcal{H}_4(f)| \leq (1/2\pi) \|f\|_{L^2}^6$. There is equality in these only if $f$ is a Gaussian. We find that for $\mathcal{H}_6$ any Gaussian gives equality, whereas for $\mathcal{H}_4$ not every Gaussian does, essentially because of the lack of a scaling law; see Proposition 5.7.

Using the representation $\mathcal{H}_6(f) = (2/\pi) \|e^{it\Delta} f\|_{L^6_t L^6_x}$, from (1.11), the $L^2$ bound on $\mathcal{H}_6$ reads

$$
\|e^{it\Delta} f\|_{L^6_t L^6_x}^6 \leq \frac{1}{2\sqrt{3}} \|f\|_{L^2}^6,
$$

which is the Strichartz inequality in dimension one. Our work shows that the constant here is the best possible, and that there is equality if and only if $f$ is a Gaussian. These facts were previously determined in [13].
We then turn to the PDE problem associated to $E_A$. Given $E_A$, a multilinear operator $T_A$ is defined implicitly by,

$$\langle T_A(f_1, \ldots, f_5), g \rangle = 2E_A(f_1, f_2, f_3, g, f_4, f_5) + 2E_A(f_1, f_2, f_3, f_4, g, f_5) + 2E_A(f_1, f_2, f_3, f_4, f_5, g).$$

From the representations (1.13) and (1.14), we find representations for the resonant equations, implicitly by,

(1.15) \[ iu_t = T_6(u, u, u, u, u) = \frac{1}{\sqrt{3\pi^2}} \int_0^{2\pi} T_R(\theta)(u, u, u, u, u)d\theta, \]

and

(1.16) \[ iu_t = T_4(u, u, u) = \frac{1}{\sqrt{2\pi^2}} \int_0^{2\pi} T_S(\theta)(G, u, u, G, u)d\theta, \]

where $R(\theta)$ and $S(\theta)$ are the same matrices as in (1.13) and (1.14).

**Theorem** (Theorem 3.7, page 18). The multilinear operator $T_A$ is bounded from $X^5$ to $X$ for (i) $X = L^2$, (ii) $X = L^{2,\sigma}$ for any $\sigma \geq 0$, and (iii) $X = H^\sigma$ for any $\sigma \geq 0$.

This theorem automatically implies analogous bounds for $T_6$ and $T_4$, and this leads directly to local wellposedness for the resonant equations in all the spaces in the theorem. By pairing this local wellposedness result with the conservation of the $L^2$ norm, we get global wellposedness in $L^2$.

**Corollary** (Theorems 4.12, page 30, and 5.9, page 44). Hamilton’s equations corresponding to $\mathcal{H}_6$ and $\mathcal{H}_4$ are locally wellposed in $X$ for (i) $X = L^2$, (ii) $X = L^{2,\sigma}$ for any $\sigma \geq 0$, and (iii) $X = H^\sigma$ for any $\sigma \geq 0$. They are globally wellposed in $L^2$.

Finally, we find that the functional $E_A$ interacts well with the Hermite functions.

**Theorem** (Theorem 3.9, page 20). Let $\{\phi_n\}_{n=0}^\infty$ be the Hermite functions. If $n_1 + n_2 + n_3 \neq n_4 + n_5 + n_6$, then $E_A(\phi_{n_1}, \phi_{n_2}, \phi_{n_3}, \phi_{n_4}, \phi_{n_5}, \phi_{n_6}) = 0$. It follows that,

$$T_A(\phi_{n_1}, \phi_{n_2}, \phi_{n_3}, \phi_{n_4}, \phi_{n_5}, \phi_{n_6}) = C\phi_{n_6},$$

for some $C$ and $n_6 = n_1 + n_2 + n_3 - n_4 - n_5$.

By using the representations of $T_6$ and $T_4$ in (1.15) and (1.16), and the fact that $G = C\phi_0$, we immediately discover that,

$$T_6(\phi_n, \phi_n, \phi_n, \phi_n, \phi_n) = C_n\phi_n \quad \text{and} \quad T_4(\phi_n, \phi_n, \phi_n) = D_n\phi_n,$$

for some constants $C_n$ and $D_n$. This immediately implies that $e^{-iC_n t}\phi_n(x)$ and $e^{-iD_n t}\phi_n(x)$ are explicit solutions of the resonant equations (1.15) and (1.16) respectively. A solution of the form $e^{i\omega t}\psi(x)$ is a stationary wave solution.

**Corollary.** For every $n \geq 0$, $\phi_n(x)$ is a stationary wave of the Hamiltonian systems $\mathcal{H}_6$ and $\mathcal{H}_4$.

By letting the symmetries of each of the equations act on $\phi_n$ we can construct more stationary waves; see (4.37) and (5.23).

1.2.4. The quintic Hamiltonian, $\mathcal{H}_6$. The previous subsection outlined results on the quintic Hamiltonian $\mathcal{H}_6$ that all arise from the representation (1.7) along with relevant properties of the functional $E_A$. Such results also apply to any composite Hamiltonian of the form,

(1.17) \[ \mathcal{H}(f) = \int_\Omega \phi(\omega)E_A(\omega)(f, \ldots, f)d\omega, \]

where $A(\omega)$ is always an isometry and $\phi$ is integrable. One of the aspirations of the present work is that other Hamiltonian systems may be cast into the framework of (1.17), and that our results on the functional $E_A$ may be applied therein.

The Hamiltonian $\mathcal{H}_6$, however, has more structure than a generic Hamiltonian of type (1.17). In Section 4 we present a number of results that are based on this additional structure and that do not follow simply from analogous properties of $E_A$. We first prove that if a stationary wave is in $L^2$, then it is automatically analytic and exponentially decaying in physical space and Fourier space.
Theorem (Corollary 4.16, page 33). Suppose that $\phi \in L^2$ is a stationary wave solution of the quintic resonant equation $(1.3)$. Then there is $\alpha, \beta > 0$ such that $\phi e^{\alpha x^2} \in L^\infty$ and $\dot{\phi} e^{\beta x^2} \in L^\infty$. In particular, $\phi$ can be extended to an analytic function on the complex plane.

We then investigate further boundedness properties of $E_0$, which lead directly to local wellposedness of Hamilton's equation in the relevant spaces. Our first result is that $T_0$ is smoothing: it maps Sobolev data to a higher Sobolev space. The second result concerns boundedness in weighted $L^\infty$ spaces.

Theorem. (i) (Theorem 4.17, page 34) For any $\sigma > 0$, there is a $\delta > 0$ and a constant $C$ such that,

$$\|T_0(f_1, f_2, f_3, f_4, f_5)\|_{L^2, \sigma + \delta} \leq C \prod_{k=1}^5 \|f_k\|_{L^{2, \sigma}} \quad \text{and} \quad \|T_0(f_1, f_2, f_3, f_4, f_5)\|_{H^{\sigma + \delta}} \leq C \prod_{k=1}^5 \|f_k\|_{H^{\sigma}}.$$

(ii) (Theorem 4.18, page 36) For any $s > 1/2$ there is a constant $C$ such that,

$$\|T_0(f_1, f_2, f_3, f_4, f_5)\|_{L^\infty, \sigma} \leq C \prod_{k=1}^5 \|f_k\|_{L^\infty, \sigma}.$$

It is expected that item (ii) here can be sharpened to show that $T_0$ is bounded from $(L^\infty)^5$ to $L^{\infty, 1/2}$ (note these are homogeneous weighted $L^\infty$ spaces). As discussed after the proof of Theorem 4.18, this is equivalent to $1/\sqrt{|x|}$ being a stationary wave of the quintic Hamiltonian system, which we conjecture.

1.2.5. The cubic Hamiltonian, $H_A$. As in the quintic case, we present a number of results on the cubic resonant equation that rely on further structure of $H_A$ beyond that given by the representation (1.16). Again, we prove that stationary waves are analytic and exponentially decaying in physical space and Fourier space once they are in $L^2$, and we examine boundedness in weighted $L^\infty$ spaces. Our stationary waves theorem, and the broad plan of the proof, are the same as those of the quintic equation, but the technical details are quite different.

Theorem (Theorem 5.13, page 48). Suppose that $\phi \in L^2$ is a stationary wave solution of the cubic resonant equation $(1.3)$. Then there is $\alpha, \beta > 0$ such that $\phi e^{\alpha x^2} \in L^\infty$ and $\dot{\phi} e^{\beta x^2} \in L^\infty$. In particular, $\phi$ can be extended to an analytic function on the complex plane.

Theorem. For any $s > 1/2$ there is a constant $C$ such that,

$$\|T_4(f_1, f_2, f_3)\|_{L^\infty, \sigma} \leq C \prod_{k=1}^3 \|f_k\|_{L^\infty, \sigma}.$$

1.3. Plan of the article. In Section 2 we prove the main approximation result. Section 3 is devoted to studying the functionals $E_A$ defined in (1.12). We study the functionals in somewhat more generality than indicated above: we assume the matrix $A$ is an isometry from $\mathbb{R}^n$ to $\mathbb{R}^n$ and, then, that $E_A$ takes $2n$ inputs. In Section 4 we present results concerning the quintic Hamiltonian system defined by $H_6$, including the representation formulas and the details of how our findings on $E_A$ translate to $E_0$. The cubic Hamiltonian system is treated in a similar fashion in Section 5. There is one appendix that deals with the technical classification of the maximizers of the $L^2$ bound on $E_A$.

1.4. Notations and conventions.

- For $x \in \mathbb{R}$, the Japanese bracket is $\langle x \rangle = \sqrt{1 + x^2}$.
- $\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$.
- The Sobolev space $H^\sigma$ is defined by the norm $\|f\|_{H^\sigma} = \|\langle x \rangle^\sigma \hat{f}\|_{L^2}$.
- The weighted space $L^{2, \sigma}$ is defined by the norm $\|f\|_{L^{2, \sigma}} = \|\langle x \rangle^\sigma f\|_{L^2}$.
- $H = -\Delta + x^2$ is the operator corresponding to the quantum harmonic oscillator.
- The Fourier transform of $f$ is $\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$. With this convention, the map $f \mapsto \hat{f}$ is an isometry of $L^2(\mathbb{R})$, and the identity $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$ holds. We will frequently use the Fourier inversion formula,

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ia \langle w, x \rangle} \phi(w)dwdx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi(aw)dw = \frac{1}{|a|} \phi(0).$$

- We set $G(x) = e^{-ax^2/2}$. For all $a > 0$, $\mathcal{F}(e^{-ax^2}) (\xi) = a^{-1/2} e^{-\xi^2/a}$ and $\int_{\mathbb{R}} e^{-ax^2}dx = \sqrt{\pi/a}$. 

• $A \preceq B$ means there is an absolute constant $C$ such that $A \leq CB$. $A \sim B$ means $A \preceq B$ and $B \preceq A$.

2. An approximation theorem

We begin the article by treating more precisely the derivation of the resonant equation (1.4) and then proving the approximation theorem described in the introduction.

Before studying the nonlinear problem, we recall some basic properties of the linear problem corresponding to (1.1). These facts will be used extensively throughout the article. The linear equation corresponding to (1.1) is simply the equation for the quantum harmonic oscillator,

\begin{equation}
\dot{u} + Hu = i u - \Delta u + x^2 u = 0,
\end{equation}

where $H = -\Delta + x^2$. For any initial data $u_0 \in L^2$ there is a unique solution to (2.1), which we denote $e^{itH}u_0$. An explicit representation of this solution is given by the Mehler formula,

\begin{equation}
e^{itH}u_0(x) = \frac{1}{\sqrt{2\pi i\sin(2t)}} \int_{\mathbb{R}} e^{-i[(x^2/2) + y^2/2]\cos(2t) - xy} \sin(2t) u_0(y) dy.
\end{equation}

(This and other properties of the linear flow may be found in [9].) From this expression we see that the solution is time-periodic with period $\pi$.

An alternative representation of the solution of (2.1) may be found by examining the Hermite functions $\{\phi_n\}_{n=0}^\infty$. The Hermite functions are eigenfunctions of $H$ – they satisfy $H\phi_n = (2n+1)\phi_n$ – and they form an orthonormal basis of $L^2$. Each of these functions is a polynomial multiplied by the Gaussian $e^{-x^2/2}$, for example,

\begin{align*}
\phi_0(x) &= c_0 e^{-x^2/2}, \\
\phi_1(x) &= c_1 x e^{-x^2/2}, \\
\phi_2(x) &= c_2 (1 - 2x^2) e^{-x^2/2},
\end{align*}

where the constants $c_n$ are normalizing constants that ensure $\|\phi_n\|_{L^2} = 1$. Using the eigenfunction property one finds that $e^{itH}\phi_n = e^{it(2n+1)}\phi_n$. Let $\Pi_n u_0 = \langle u_0, \phi_n \rangle \phi_n$ be the orthogonal projection onto the eigenspace spanned by $\phi_n$. Given any $u_0 \in L^2$ we may expand $u_0(x) = \sum_{n=0}^{\infty} (\Pi_n u_0)(x)$, and then find,

\begin{equation}
e^{itH}u_0(x) = \sum_{n=0}^{\infty} e^{it(2n+1)} (\Pi_n u_0)(x),
\end{equation}

so the flow has a simple description in the Hermite function coordinates. We finally note that the Hermite functions satisfy $\phi_n(-x) = (-1)^n \phi_n(x)$, as may be inferred from the formula $\phi_n(x) = c_n e^{-x^2/2} (d^n/dx^n) e^{-x^2}$ from [9].

We now turn to the nonlinear problem (1.1). The linear part of the equation may be absorbed into the nonlinearity by changing variables to the profile $v(x,t) = e^{-itH}u(x,t)$. The function $v$ satisfies the equation,

\begin{equation}
iv_t = e^{-itH} \left( (e^{itH} v)^{2k} e^{itH} v \right) := N_t(v, \ldots, v),
\end{equation}

where $N_t$ is the $(2k+1)$ multilinear functional,

\begin{equation}
N_t(f_1, \ldots, f_{2k+1}) = e^{-itH} \left( \prod_{m=1}^{k} (e^{itH} f_m)(e^{itH} f_{k+1+m}) \right) (e^{itH} f_{k+1}).
\end{equation}

We expand each of the functions $f_m$ in the basis of Hermite functions,

\begin{equation}
e^{itH} f_m = e^{itH} \left( \sum_{n_m=0}^{\infty} \Pi_{n_m} f_m \right) = \sum_{n_m=0}^{\infty} e^{it(2n_m+1)} \Pi_{n_m} f_m,
\end{equation}

and then substitute into (2.4). This yields,

\begin{equation}
N_t(f_1, \ldots, f_{2k+1}) = \sum_{n_{1}, \ldots, n_{2k+2} \geq 0} e^{2tL \Pi_{n_{2k+2}}} \left[ \prod_{m=1}^{k} (\Pi_{n_m} f_m)(\Pi_{n_{k+1+m}} f_{k+1+m}) \right] \Pi_{n_{k+1}} f_{k+1},
\end{equation}

where $L = \sum_{m=1}^{k+1} n_m - n_{k+m+1}$. 

In (2.5), when $L \neq 0$ the associated term in the sum is oscillating, while when $L = 0$ the associated term is not. The resonant equation arises simply from neglecting the oscillatory terms. Define the multilinear functional $T$ by

$$
T(f_1, \ldots, f_{2k+1}) = \sum_{n_1, \ldots, n_{2k+2} \geq 0} \prod_{L=0}^k \left( \prod_{m=1}^k (\Pi_{n_m} f_m) (\Pi_{n_{k+1+m}} f_{k+1+m}) \right) \Pi_{n_{k+1}} f_{k+1}.
$$

The resonant PDE is then given by,

$$
iw_t = T(w, \ldots, w).
$$

**Lemma 2.1.** The resonant functional $T$ is the time average of the functionals $N_r$ over the interval $[-\pi/4, \pi/4]$; that is,

$$
T(f_1, \ldots, f_{2k+1}) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} N_r(f_1, \ldots, f_{2k+1}) dr.
$$

**Proof.** We integrate the sum in (2.5) over $[-\pi/4, \pi/4]$ term by term. If $L = 0$ nothing changes and we get the associated term in (2.6). If $L$ is even then $\int_{-\pi/4}^{\pi/4} e^{2iLr} dr = 0$, and the term in (2.5) is 0. Finally if $L$ is odd, then either $n_{2k+2}$ is even and $L - n_{2k+2}$ is odd, or $n_{2k+2}$ is odd and $L - n_{2k+2}$ is even. In the first case we have, using the Hermite function property $(\Pi_n f)(-x) = (-1)^n (\Pi_n f)(x)$, that,

$$
\left( \prod_{m=1}^k ((\Pi_{n_m} f_m)(-x)) (\Pi_{n_{k+1+m}} f_{k+1+m})(-x) \right) (\Pi_{n_{k+1}} f_{k+1})(x),
$$

and hence the function here is odd. Projecting onto the eigenspace spanned by the even function $\phi_{n_{2k+2}}$ gives the 0 vector. The associated term in the sum (2.5) is thus 0. In the case when $n_{2k+2}$ is even and $L - n_{2k+2}$ is odd a similar analysis shows that the term in the sum is again 0. In conclusion, all of terms corresponding to $L \neq 0$ vanish, while those corresponding to $L = 0$ are unchanged. \qed

By virtue of the lemma the resonant equation can be written as,

$$
iw_t = T(w, \ldots, w) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} N_r(w(t), \ldots, w(t)) dr = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} e^{-irH} (|e^{irH} w(t)|^{2k} e^{irH} w(t)) dr,
$$

which is precisely (1.4). One can show that the resonant equation is the flow corresponding the Hamiltonian,

$$
H_{2k+2}(f) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} |e^{irH} f(x)|^{2k+2} dx dr.
$$

The details of this Hamiltonian correspondence a presented in Theorem 4.1 below.

We now prove the approximation theorem. The theorem is essentially a lower dimensional analog of Theorem 3.1 in [17], and our proof follows theirs closely. The function space in our theorem is

$$
\mathcal{H}^s = \{ u \in L^2 : H^{s/2} u \in L^2 \},
$$

with the norm $\| u \|_{\mathcal{H}^s} = \| H^{s/2} u \|_{L^2}$. From [26], we have the norm equivalence $\| u \|_{\mathcal{H}^s} \sim \| (x)^{s/2} u \|_{L^2} + \| (\xi)^{s/2} \hat{u} \|_{L^2}$. This space $\mathcal{H}^s$ is useful for two reasons: first, if $s > 1/2$, then the space is an algebra (as a direct consequence of the norm equivalence); and, second, the space interacts well with the linear propagator $e^{itH}$, as seen in the following Lemma.

**Lemma 2.2.** Fix $s \geq 0$. For all $u \in \mathcal{H}^s$ and $t \in \mathbb{R}$ we have $\| e^{itH} u \|_{\mathcal{H}^s} \leq \| u \|_{\mathcal{H}^s}$.

A general $L^p$ version of this lemma appears in [4]; for $L^2$, there is the following shorter proof.
Proof. First let $s$ be an even non-negative integer. Write $u \in L^2$ in the basis of Hermite functions as $u = \sum_{n=0}^{\infty} a_n \phi_n$. Because $s$ is a non-negative even integer, for every $n$ we have $H^{s/2} \phi_n = (2n+1)^{s/2} \phi_n$. This then gives,

$$\|e^{itH}u\|_{H^s}^2 = \left\| H^{s/2} \sum_{n=0}^{\infty} a_n e^{it(2n+1)} \phi_n \right\|_{L^2}^2 = \left\| \sum_{n=0}^{\infty} a_n e^{it(2n+1)}(2n+1)^{s/2} \phi_n \right\|_{L^2}^2$$

$$= \sum_{n=0}^{\infty} |a_n e^{it(2n+1)}(2n+1)^{s/2} \phi_n|_{L^2}^2 \quad \text{(by orthogonality)}$$

$$= \sum_{n=0}^{\infty} |a_n (2n+1)^{s/2} | \phi_n |_{L^2}^2 = \|u\|_{H^s}^2.$$

The result for general $s$ follows from interpolation. □

**Theorem 2.3.** Fix $s > 1/2$ and initial data $u_0 \in H^s$. Let $u$ be a solution of the nonlinear Schrödinger equation with harmonic trapping (1.1) and $w$ a solution of the resonant equation (2.7), both corresponding to the initial data $u_0$. Suppose that the bounds $\|u(t)\|_{H^r}, \|w(t)\|_{H^r} \leq \epsilon$ hold for all $t \in [0,T]$. Then for all $t \in [0,T]$,

$$\|u(t) - e^{itH}w(t)\|_{H^r} \leq \left( t(2k+1)e^{4k+1} + e^{2k+1} \right) \exp \left( (2k+1)t e^{2k} \right).$$

In particular if $t \lesssim e^{-2k}$ then $\|u(t) - e^{-itH}w(t)\|_{H^r} \lesssim e^{2k+1}$.

**Proof.** Let $v(x,t) = e^{-itH}u(x,t)$, so that $v$ satisfies the PDE (2.3). We note that $v(x,0) = u(x,0) = u_0(x)$. Using the lemma, we find that,

$$\|u(t) - e^{itH}w(t)\|_{H^r} = \|e^{itH}v(t) - e^{itH}w(t)\|_{H^r} \leq \|v(t) - w(t)\|_{H^r}.$$

To prove the theorem it therefore suffices to show that $v$ and $w$ are close in $H^s$.

Therefore let $v$ and $w$ be solutions of the equations (2.3) and (2.7) respectively with the same initial data $u_0$,

$$iv_t(t) = N_i(v(t), \ldots, v(t)) = e^{-itH} \left( |e^{itH}v(t)|^{2k} e^{itH}v(t) \right),$$

$$iw_t(t) = T(w(t), \ldots, w(t)) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} e^{-irH} \left( |e^{irH}w(t)|^{2k} e^{irH}w(t) \right) dr,$$

$$u_0(x) = v(x,0) = u(x,0).$$

Set,

$$D_t(f_1, \ldots, f_{2k+1}) = N_t(f_1, \ldots, f_{2k+1}) - T(f_1, \ldots, f_{2k+1})$$

$$= \sum_{n_1, \ldots, n_{2k+2} \geq 0, \ L \neq 0} e^{2itH} \Pi_{2k+2} \left( \prod_{m=1}^{k} (\Pi_{n_m} f_m) (\Pi_{nk+1+m} f_{k+1+m}) \right) \Pi_{nk+1} f_{k+1}.$$

From the expressions of the multilinear operators $N_t$ and $T$ in (2.11) and (2.12) (or their multilinear versions (2.4) and (2.8)), from Lemma 2.2, and from the fact that $H^s$ is an algebra, it follows that $N_t$ and $T$ are uniformly bounded from $(H^r)^{2k+1}$ to $H^s$. The same holds for $D_t$ from (2.14).

Set $\phi(t) = v(t) - w(t)$. Because $\phi(0) = 0$, the Duhamel form of the equation on $\phi$ is,

$$i\phi(t) = \int_0^t [T(v(r), \ldots, v(r)) - T(w(r), \ldots, w(r)) + D_v(v(r), \ldots, v(r))] dr$$
We will determine \( a \) priori \( bounds \) on \( \phi \). For the first term in the integrand here, we can expand by multilinearity to find,

\[
(2.17) \quad \| T(v(r), \ldots, v(r)) - T(w(r), \ldots, w(r)) \|_{H^s} \leq \sum_{m=0}^{2k} \| T(v(r), \ldots, v(r), v(r) - w(r), w(r), \ldots, w(r)) \|_{H^s}^{2k-m}.
\]

\[
(2.18) \quad \leq \sum_{m=0}^{2k} \| v(r) \|_{H^s}^{m} \| v(r) - w(r) \|_{H^s} \| w(r) \|_{H^s}^{2k-m}.
\]

\[
(2.19) \quad \leq (2k + 1) \epsilon^{2k} \| v(r) - w(r) \|_{H^s}.
\]

For the second term in the integrand in (2.16) we need to look more closely at the operator \( D_t \). We first observe the identity,

\[
e^{2i\theta L} = \frac{d}{ds} \int_{\theta}^{\theta + 2\pi L} e^{2i\beta L} d\theta,
\]

where \( \lfloor x \rfloor \) is the smallest integer less than \( x \). (Recall from the proof of the first lemma that only even values of \( L \) contribute to the sum in (2.15).) The interval of integration here has length less than 1. We can then handle the second term in (2.16) as follows,

\[
\int_0^t [D_r(v(r), \ldots, v(r))] ds = \sum_{n_1, \ldots, n_{2k+2} \geq 0 \atop L \neq 0} \int_0^t \int_{\frac{x}{2\pi}}^{\frac{2x}{\pi}} e^{2i\theta L} d\theta \Pi_{n \geq 2} \left[ \left( \prod_{m=1}^k (\Pi_{n_m} v(r)) \Pi_{n \geq k+1} v(r) \right) \Pi_{n \geq k+1} v(r) \right] dr.
\]

Using integration by parts, we have

(\text{left hand side})

\[
= - \sum_{n_1, \ldots, n_{2k+2} \geq 0 \atop L \neq 0} \int_0^t \left( \int_{\frac{x}{2\pi}}^{\frac{2x}{\pi}} e^{2i\theta L} d\theta \right) \Pi_{n \geq 2} \left[ \left( \prod_{m=1}^k (\Pi_{n_m} v(r)) \Pi_{n \geq k+1} v(r) \right) \Pi_{n \geq k+1} v(r) \right] dr
\]

\[
+ \sum_{n_1, \ldots, n_{2k+2} \geq 0 \atop L \neq 0} \left( \int_{\frac{x}{2\pi}}^{\frac{2x}{\pi}} e^{2i\theta L} d\theta \right) \Pi_{n \geq 2} \left[ \left( \prod_{m=1}^k (\Pi_{n_m} v(t)) \Pi_{n \geq k+1} v(t) \right) \Pi_{n \geq k+1} v(t) \right]
\]

\[
= - \int_0^t \sum_{m=0}^{2k} \int_{\frac{x}{2\pi}}^{\frac{2x}{\pi}} D_\theta(v(r), \ldots, v(r), v_r(r), v(r), \ldots, v(r)) ds dr
\]

\[
+ \left( \int_{\frac{x}{2\pi}}^{\frac{2x}{\pi}} D_\theta(v(t), \ldots, v(t)) ds \right).
\]

Because the interval of integration \( \left[ \frac{x}{2\pi}, \frac{2x}{\pi} \right] \) has length less than 1, we get,

\[
(2.20) \quad \left\| \int_0^t [D_r(v(r), \ldots, v(r))] ds \right\|_{H^s} \leq t(2k + 1) \sup_{r \in [0,t]} (\| v(r) \|_{H^s}^{2k} \| v_r(r) \|_{H^s} + \| v(t) \|_{H^s}^{2k+1})
\]

\[
\leq t(2k + 1) \epsilon^{4k+1} + \epsilon^{2k+1},
\]

where in the last line we have used \( \| v \|_{H^s} \leq \| v \|_{H^s}^{2k+1} \leq \epsilon^{2k+1}, \) coming from (2.3).
Combining the estimates (2.19) and (2.20) we get
\[ \|\phi(t)\|_{H^r} \leq (2k+1)^{2^k} \int_0^t \|\phi(s)\|_{H^r} ds + t(2k+1)\epsilon^{4k+1} + \epsilon^{2k+1}. \]

Gronwell’s inequality then implies that,
\[ \|v(t) - w(t)\|_{H^r} = \|\phi(t)\|_{H^r} \leq (t(2k+1)\epsilon^{4k+1} + \epsilon^{2k+1}) \exp ((2k+1)t\epsilon^{2k}), \]

which with (2.10) gives the theorem. \( \square \)

3. Analysis of a class of multilinear functionals

In the introduction we presented two formulas (1.7) and (1.9) that represent the Hamiltonians \( H_6 \) and \( H_4 \) in terms of simpler functionals of the form \( E_A \).

\[ H_6(f) = \frac{1}{2\sqrt{3\pi}} \int_0^{2\pi} E_{R(\theta)}(f, f, f, f, f, f) d\theta, \quad H_4(f) = \frac{1}{2\sqrt{2\pi^2}} \int_0^{2\pi} E_{S(\theta)}(G, f, f, G, f, f) d\theta. \]

Here, for an isometry \( A : \mathbb{R}^n \to \mathbb{R}^n \), the 2n multilinear functional \( E_A \) is defined by,

\[ E_A(f_1, \ldots, f_{2n}) = \int_{\mathbb{R}^n} \prod_{k=1}^n f_k((Ax)_k) \bar{f}_{n+k}(x_k) dx, \]

where \( x \in \mathbb{R}^n, x_k = \langle x, e_k \rangle \) and \( (Ax)_k = \langle Ax, e_k \rangle \). (In the introduction, and in formulas (3.1), \( n \) is set to 3, but the work in this section is for arbitrary \( n \).) As stated in the introduction, it turns out that one can gain significant insight into the dynamics of the systems associated to the Hamiltonians \( H_6 \) and \( H_4 \) by understanding properties of the functionals \( E_A \). This section, therefore, is a general study of this family of functionals. We will examine the symmetries of \( E_A \), its boundedness in \( L^2 \) and higher Sobolev spaces, and its relationship to the Hermite functions. Our motivation throughout is to relate these findings back to the dynamics of the Hamiltonian systems defined by \( H_6 \) and \( H_4 \). For this reason we will also present a number of results that relate properties of a generic multilinear functional \( E(f_1, \ldots, f_{2n}) \), to the flow induced by the Hamiltonian \( H(f) = E(f, \ldots, f) \) associated to it.

Our approach here is abstract, but we consider the abstraction justified for three reasons. First, it is efficient. Once we have proved, for example, \( L^2 \) local wellposedness for the partial differential equation induced by (3.2), it will immediately imply \( L^2 \) local wellposedness for the two distinct systems defined by (3.1). Second, our approach clarifies which structure in the Hamiltonians (3.1) is responsible for certain dynamics. As a byproduct, it suggests that many of the remarkable properties of these Hamiltonian systems (large number of symmetries, wellposedness in many spaces, Hermite functions as stationary waves) are generic. Third, one might expect that there are other Hamiltonian systems of mathematical or physical interest that can be cast into the framework suggested by the representations in (3.1). Our results here would immediately give significant insight into the dynamics of such systems.

3.1. The multilinear functional framework. In what follows, plain Latin letters such as \( E_A \), \( T_A \) and \( H_A \) will denote the specific multilinear functional defined by (3.2) and objects associated to it. Curly letters \( \mathcal{E}, \mathcal{T}, \) and \( \mathcal{H} \) will denote a generic multilinear functional, multilinear operator and Hamiltonian respectively. All multilinear functionals take 2n arguments, are linear in the first \( n \) arguments and conjugate linear in the last \( n \) arguments, as in (3.2).

The functional properties of \( E_A \) depend strongly on the matrix properties of \( A \). In the representations in (3.1), the matrices involved are all isometries, and we will find that this structural property plays a key role in the analysis. We will therefore assume throughout that the matrix \( A \) is an isometry.

**Definition 3.1.** (i) To each matrix \( A \) we associate a multilinear operator \( T_A \) defined implicitly by the formula,

\[ (T_A(f_1, \ldots, f_{2n-1}), g)_{L^2} = 2 \sum_{k=1}^n E_A(f_1, \ldots, f_{n+k-1}, g, f_{n+k}, \ldots, f_{2n-1}). \]

(ii) To each matrix \( A \) we associate a function \( H_A \) defined by \( H_A(f) = E_A(f, \ldots, f) \).

These definitions are motivated by the following theorem.
Theorem 3.1. Suppose that a multilinear functional \( \mathcal{E}(f_1, \ldots, f_{2n}) \) has the permutation symmetry,

\[
\mathcal{E}(f_1, \ldots, f_n, f_{n+1}, \ldots, f_{2n}) = \mathcal{E}(f_{n+1}, \ldots, f_{2n}, f_1, \ldots, f_n).
\]

Then \( \mathcal{H}(f) = \mathcal{E}(f, \ldots, f) \) is a real valued function and hence a Hamiltonian on the phase space \( L^2(\mathbb{R} \to \mathbb{C}) \). Hamilton’s equation of motion is given by \( iu_t(t) = \mathcal{T}(u(t), \ldots, u(t)) \) where the multilinear operator \( \mathcal{T} \) is defined implicitly by

\[
\langle \mathcal{T}(f_1, \ldots, f_{2n-1}), g \rangle_{L^2} = 2 \sum_{k=1}^{n} \mathcal{E}(f_1, \ldots, f_{n+k-1}, g, f_{n+k}, \ldots, f_{2n-1}).
\]

Proof. First, if (3.4) is satisfied, then \( \mathcal{H}(f) = \mathcal{E}(f, \ldots, f) = \mathcal{E}(f, \ldots, f) = \overline{\mathcal{H}(f)} \), and therefore \( \mathcal{H}(f) \in \mathbb{R} \).

In order to find Hamilton’s equation of motion corresponding to \( \mathcal{H} \), we first recall the Hamiltonian phase space structure of \( L^2(\mathbb{R} \to \mathbb{C}) \). A symplectic form on \( L^2 \) is given by \( \omega(f, g) = -\text{Im}(f, g)_{L^2} \). Given a Hamiltonian \( \mathcal{H} : L^2 \to \mathbb{R} \), the symplectic gradient \( \nabla_{\omega} \mathcal{H} \) is defined as the unique solution of the equation

\[
\omega(\nabla_{\omega} \mathcal{H}(f), g) = \frac{d}{de} \bigg|_{e=0} \mathcal{H}(f + eg).
\]

Hamilton’s equation is then \( u_t = \nabla_{\omega} \mathcal{H}(u) \).

In the case when \( \mathcal{H}(f) = \mathcal{E}(f, \ldots, f) \), we have, by multilinearity,

\[
\frac{d}{de} \bigg|_{e=0} \mathcal{H}(f + eg) = \frac{d}{de} \bigg|_{e=0} \mathcal{E}(f + eg, \ldots, f + eg)
\]

\[
= 2 \sum_{k=1}^{n} \mathcal{E}(f, \ldots, f, g, f, \ldots, f) = 2 \text{Re} \sum_{k=1}^{n} \mathcal{E}(f, \ldots, f, g, f, \ldots, f),
\]

where in the last step we used the permutation symmetry (3.4). On the other hand, setting \( i\nabla_{\omega} \mathcal{H}(f) = \mathcal{T}(f, \ldots, f) \), we find,

\[
\omega(\nabla_{\omega} \mathcal{H}(f), g) = -\text{Im}(i\mathcal{T}(f, \ldots, f), g) = \text{Re}(\mathcal{T}(f, \ldots, f), g).
\]

By the definition of the symplectic gradient in (3.6), the right hand sides of (3.7) and (3.8) must match for all \( f \) and \( g \). By replacing \( g \) by \( ig \) and using conjugate linearity, we see that this equality condition holding for all \( g \) actually implies that,

\[
\langle \mathcal{T}(f, \ldots, f), g \rangle = 2 \sum_{k=1}^{n} \mathcal{E}(f, \ldots, f, g, f, \ldots, f),
\]

which, in polarized form, is precisely (3.5).

Finally, Hamilton’s equation is \( iu_t = i\nabla_{\omega} \mathcal{H}(u) = \mathcal{T}(u, \ldots, u) \). \( \Box \)

For a generic isometry \( A \), the functional \( E_A \) does not satisfy the permutation symmetry condition (3.4). However, if we define, for example,

\[
\tilde{E}_A(f_1, \ldots, f_{2n}) = \frac{1}{2} \left[ E_A(f_1, \ldots, f_{2n}) + \overline{E_A(f_{n+1}, \ldots, f_{2n}, f_1, \ldots, f_n)} \right],
\]

then \( \tilde{E}_A \) does satisfy (3.4), and all the properties of \( E_A \) we prove below carry over to \( \tilde{E}_A \). We will not be concerned with this point, because while the functionals \( E_A \) do not have the permutation symmetry (3.4), the functionals \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) defined in (1.8) and (1.10) do.

Before presenting general results on \( E_A \), we give two concrete examples. These two examples illustrate how different isometries \( A \) can give rise to very different partial differential equations \( iu_t = \mathcal{T}_A(u, \ldots, u) \).
Example 3.1. Take $n = 2$ and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity matrix. Then,
\[ E_A(f_1, f_2, f_3, f_4) = \int_{\mathbb{R}^2} f_1(x_1) f_2(x_2) f_3(x_1) f_4(x_2) dx = (f_1, f_3)_{L^2} (f_2, f_4)_{L^2}, \]
and hence $H_A(f) = \|f\|_{L^2}$. We calculate,
\[ T_A(f_1, f_2, f_3)(y) = 2 \left( \int_{\mathbb{R}} f_1(x_1) \left[ f_2(x_2) f_3(x_2) \right] dx + \int_{\mathbb{R}} f_1(x_1) f_3(x_1) f_2(x_2) dx \right) \]
\[ = 2 f_1(y) (f_2, f_3)_{L^2} + 2 f_2(y) (f_1, f_3)_{L^2}. \]
Hamilton’s equation is then $iu_t = T_A(u, u, u) = 4u\|u\|_{L^2}^2$, which has a unique solution for initial data $u_0 \in L^2$ given by $u(x, t) = e^{4i\|u_0\|_{L^2}^2 t} u_0(x)$.

Example 3.2. Take $n = 2$ again, and let $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ be the rotation of $\mathbb{R}^2$ by $\pi/4$ radians. Then,
\[ E_A(f_1, f_2, f_3, f_4) = \int_{\mathbb{R}^2} f_1 \left( \frac{x_1 - x_2}{\sqrt{2}} \right) f_2 \left( \frac{x_1 + x_2}{\sqrt{2}} \right) f_3(x_1) f_4(x_2) dx. \]
and,
\[ (3.9) \quad T_A(f_1, f_2, f_3)(y) = 2 \left( \int_{\mathbb{R}} f_1 \left( \frac{y - s}{\sqrt{2}} \right) f_2 \left( \frac{y + s}{\sqrt{2}} \right) f_3(s) ds + \int_{\mathbb{R}} f_1 \left( \frac{s - y}{\sqrt{2}} \right) f_2 \left( \frac{s + y}{\sqrt{2}} \right) f_3(s) ds \right). \]
In this case it is not clear that the general solution of the equation $iu_t = T_A(u, u, u)$ can be written explicitly. However it is still possible to determine many properties of the flow. For example, one may verify by substitution that, for any $\alpha > 0$, the functions $u(x, t) = e^{\sqrt{8\pi/\alpha} t} e^{-\alpha x^2}$ and $u(x, t) = xe^{-\alpha x^2}$, are explicit solutions of $iu_t = T_A(u, u, u)$ (the second solution does not depend on time). These solutions were both produced using Corollary 3.10 below.

3.2. Symmetries of the functional and associated conservation laws. In this subsection we uncover some of the rich symmetry structure of the functional $E_A$. We recall that $A$ is assumed to be an isometry throughout.

Theorem 3.2. The functional $E_A$ is invariant under the Fourier transform, that is,
\[ E_A(\hat{f}_1, \ldots, \hat{f}_{2n}) = E_A(f_1, \ldots, f_{2n}). \]
It follows that $T_A(\hat{f}_1, \ldots, \hat{f}_{2n-1})(\xi) = \hat{T}_A(f_1, \ldots, f_{2n-1})(\xi)$.

Proof. Because $A$ is an isometry, we have $\langle \xi, Ax \rangle = \langle A^{-1} \xi, x \rangle$ for all $\xi, x \in \mathbb{R}^n$. Now calculating,
\[ E_A(\hat{f}_1, \ldots, \hat{f}_{2n}) = \int_{\mathbb{R}^n} \prod_{k=1}^n \hat{f}_k(\langle Ax \rangle k) \overline{\hat{f}_{n+k}(x_k)} dx \]
\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \prod_{k=1}^n \left( \int_{\mathbb{R}} e^{-i\xi_k \langle Ax \rangle k} f_k(\xi_k) d\xi_k \right) \left( \int_{\mathbb{R}} e^{i\nu_k x_k} \overline{\hat{f}_{n+k}(\nu_k)} d\nu_k \right) dx \]
\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(A^{-1} \xi - \nu, x)} \prod_{k=1}^n f_k(\xi_k) \overline{\hat{f}_{n+k}(\nu_k)} d\xi d\nu dx, \]
where in the last line we have used $\langle \xi, Ax \rangle + \langle \nu, x \rangle = \langle A^{-1} \xi - \nu, x \rangle$. We first change variables $y(\xi) = A^{-1} \xi - \nu$, or $\xi(y) = Ay + Av$. The determinant of this change of variables is 1 because $A$ is an isometry. Performing the change of variables then gives the required identity,
\[ E_A(\hat{f}_1, \ldots, \hat{f}_{2n}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(y, x)} \prod_{k=1}^n f_k(Av_k + Ay) \overline{\hat{f}_{n+k}(\nu_k)} dy d\nu dx \]
\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{k=1}^n f_k(\langle Av \rangle k) \overline{\hat{f}_{n+k}(\nu_k)} d\nu = E_A(f_1, \ldots, f_{2n}), \]
where in the second equality we used the Fourier inversion identity (1.18) with $a = 1$. ✔
For the operator statement, let $g$ be an arbitrary element of $L^2$. Using the definition of $T_A$ in (3.3) we have,

$$
(T_A(f_1, \ldots, f_{2n-1}), \hat{g}) = (T_A(f_1, \ldots, f_{2n-1}), g) = 2 \sum_{k=1}^{n} E_A(f_1, \ldots, f_{n+k-1}, g, f_{n+k}, \ldots, f_{2n-1})
$$

(3.11)

$$
= 2 \sum_{k=1}^{n} E_A(\hat{f}_1, \ldots, \hat{f}_{n+k-1}, \hat{g}, \hat{f}_{n+k}, \ldots, \hat{f}_{2n-1}) = (T_A(\hat{f}_1, \ldots, \hat{f}_{2n-1}), \hat{g}).
$$

The operator identity follows.

**Theorem 3.3.** The functional $E_A$ is invariant under the following actions (for any $\lambda$):

(i) Modulation: $f_k \mapsto e^{i\lambda} f_k$.

(ii) $L^2$ scaling: $f_k(x) \mapsto \lambda^{1/2} f_k(\lambda x)$.

(iii) Quadratic modulation: $f_k \mapsto e^{i\lambda |x|^2} f_k$.

(iv) Schrödinger group: $f_k \mapsto e^{i\Delta t} f_k$.

(v) Schrödinger with harmonic trapping group: $f_k \mapsto e^{i\lambda H} f_k$, where $H = -\Delta + |x|^2$.

If, in addition, $A$ satisfies $Ae = e$, where $e = (1, \ldots, 1) \in \mathbb{R}^n$, then $E_A$ is invariant under the following actions (for any $\lambda$):

(vi) Linear modulation: $f_k \mapsto e^{i\lambda x} f_k$.

(vii) Translation: $f_k \mapsto f_k(\cdot + \lambda)$.

**Proof.**

(i) We have,

$$
E_A(e^{i\lambda} f_1, \ldots, e^{i\lambda} f_{2n}) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} e^{i\lambda f_1((Ax)_k)} e^{-i\lambda} \mathcal{T}_{2n}(x_k) dx = E_A(f_1, \ldots, f_{2n}).
$$

(ii) Let $f_k^\lambda(x) = \lambda^{1/2} f_k(\lambda x)$. We write out $E_A$ and perform the change of variables $y = \lambda x$ (with $dy = \lambda^n dx$) to find,

$$
E_A(f_1^\lambda, \ldots, f_{2n}^\lambda) = \lambda^n \int_{\mathbb{R}^n} \prod_{k=1}^{n} f_k(\lambda |Ax|_k) \mathcal{T}_{n+k}(\lambda x_k) dx = \int_{\mathbb{R}^n} \prod_{k=1}^{n} f((Ay)_k) \mathcal{T}(x_k) dx = E_A(f_1, \ldots, f_{2n}).
$$

(iii) Because $A$ is an isometry, $|Ax|^2 = |x|^2$ for all $x \in \mathbb{R}^n$. Using this, we have,

$$
E_A(e^{i|\lambda x|^2} f_1, \ldots, e^{i|\lambda x|^2} f_{2n}) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} e^{i|\lambda(Ax)_k|^2} f_k((Ax)_k) e^{-i\lambda |x_k|^2} \mathcal{T}_{n+k}(x_k) dx
$$

$$
= \int_{\mathbb{R}^n} e^{i\lambda |Ax|^2} e^{-i|\lambda x|^2} \prod_{k=1}^{n} f_k((Ax)_k) \mathcal{T}_{n+k}(x_k) dx = E_A(f_1, \ldots, f_{2n}).
$$

(iv) Using the previous part and the invariance of the Hamiltonian under the Fourier transform, we find,

$$
E_A(e^{i\Delta} f_1, \ldots, e^{i\Delta} f_{2n}) = E_A(e^{-i|\lambda x|^2} f_1, \ldots, e^{-i|\lambda x|^2} f_{2n}) = E_A(\hat{f}_1, \ldots, \hat{f}_{2n}) = E_A(f_1, \ldots, f_{2n}).
$$

(v) In this part we use $t$ instead of $\lambda$, and show invariance of the functional under $e^{itH}$.

First, we note that if $n$ is an integer then $e^{i(\pi/2 + n\pi)t} f = \hat{f}$ (from, for instance, the Mehler formula (2.2)). The $t = \pi/2 + n\pi$ case thus follows from Theorem 3.2.

If $t \neq \pi/2 + n\pi$ then we may represent $e^{itH} f$ using the lens transform [24]. This transform relates solutions of the free linear Schrödinger to the linear Schrödinger equation with harmonic trapping. Precisely, there holds,

$$
(e^{itH} f_k)(x) = \frac{1}{\sqrt{\cos(2t)}} (e^{i(t\tan(2t)/2)\Delta} f_k) \left( \frac{x}{\cos(2t)} \right) e^{ix^2 \tan(2t)/2}.
$$

(3.12)
We substitute this expression into the functional. Using in turn the symmetries (iii) (with \(\lambda = \tan(2t)/2\)), (ii) (with \(\lambda = 1/\cos(2t)\)), and (iv) (with \(\lambda = \tan(2t)/2\)), we determine that,

\[
E_{A}(e^{itH}f_{1},\ldots,e^{itH}f_{2n})
\]

\[
= E_{A} \left( \frac{1}{\cos(2t)}e^{it\frac{1}{\cos(2t)}\Delta}f_{1}\left(\frac{x}{\cos(2t)}\right),\ldots,\frac{1}{\cos(2t)}e^{it\frac{1}{\cos(2t)}\Delta}f_{2n}\left(\frac{x}{\cos(2t)}\right) \right)
\]

\[
= E_{A} \left( (e^{it\frac{1}{\cos(2t)}\Delta}f_{1}) (x),\ldots,(e^{it\frac{1}{\cos(2t)}\Delta}f_{2n}) (x) \right) = E_{A}(f_{1},\ldots,f_{2n}).
\]

(vi) In these last two parts we assume that, in addition to being an isometry, the matrix \(A\) also satisfies \(Ac = e\) for \(e = (1,\ldots,1) \in \mathbb{R}^{n}\). We then have,

\[
E_{A}(e^{i\lambda x}f_{1},\ldots,e^{i\lambda x}f_{2n}) = \int_{\mathbb{R}^{n}} \prod_{k=1}^{n} e^{i\lambda(Ax)_{k}}f_{k}((Ax)_{k})e^{-i\lambda x_{k}} f_{n+k}(x_{k})dx
\]

\[
= \int_{\mathbb{R}^{n}} e^{i\lambda(Ax,e)}e^{-\lambda(x,e)} \prod_{k=1}^{n} f_{k}((Ax)_{k}) f_{n+k}(x_{k})dx = E_{A}(f_{1},\ldots,f_{2n}).
\]

where in the last step we used \(\langle Ax,e \rangle = (x,A^{-1}e) = (x,e)\).

(vii) This follows immediately from the previous part and the invariance of the functional under the Fourier transform, as in item (iv), noting that the Fourier transform takes \(x \mapsto e^{i\lambda x}f(x)\) to \(\xi \mapsto \hat{f}(\xi + \lambda)\).

\[\square\]

The symmetries of the functional \(E_{A}\) lead directly to commutator identities for the operator \(T_{A}\).

**Corollary 3.4.** We have the following commutator identities,

\[
e^{i\lambda Q}T_{A}(f_{1},\ldots,f_{2n-1}) = T_{A}(e^{i\lambda Q}f_{1},\ldots,e^{i\lambda Q}f_{2n-1})
\]

\[
QT_{A}(f_{1},\ldots,f_{2n-1}) = \sum_{k=1}^{n} T_{A}(f_{1},\ldots,f_{k-1},Qf_{k},f_{k+1},\ldots,f_{2n-1})
\]

\[
= \sum_{k=n+1}^{2n-1} T_{A}(f_{1},\ldots,f_{k-1},Qf_{k},f_{k+1},\ldots,f_{2n-1})
\]

(3.14)

where \(Q\) is any of the following operators.

1. For a generic isometry \(A\), \(Q = x^{2}\), \(Q = \Delta\) and \(Q = H\).
2. If in addition \(Ac = e\), where \(e = (1,\ldots,1)\), \(Q = x\), \(Q = id/dx\).

**Proof.** For each of the operators \(Q\), the flow map \(e^{i\lambda Q}\) is an isometry of \(L^{2}\) for all \(\lambda\), and,

\[
E_{A}(e^{i\lambda Q}f_{1},\ldots,e^{i\lambda Q}f_{2n}) = E_{A}(f_{1},\ldots,f_{2n}),
\]

from Theorem 3.3. For each \(g \in L^{2}\), we thus have,

\[
\langle e^{i\lambda Q}T_{A}(f_{1},\ldots,f_{2n-1}),g \rangle_{L^{2}} = \langle T_{A}(f_{1},\ldots,f_{2n-1}),e^{-i\lambda Q}g \rangle_{L^{2}}
\]

\[
= 2 \sum_{k=1}^{n} E_{A} \left( f_{1},\ldots,f_{n+k-1},e^{-i\lambda Q}g,f_{n+k},\ldots,f_{2n-1} \right)
\]

\[
= 2 \sum_{k=1}^{n} E_{A} \left( e^{i\lambda Q}f_{1},\ldots,e^{i\lambda Q}f_{n+k-1},g,e^{i\lambda Q}f_{n+k},\ldots,e^{i\lambda Q}f_{2n-1} \right)
\]

\[
= \langle T_{A}(e^{i\lambda Q}f_{1},\ldots,e^{i\lambda Q}f_{2n-1}),g \rangle_{L^{2}},
\]

which gives (3.13). To get (3.14), differentiate (3.13) with respect to \(\lambda\) and set \(\lambda = 0\).

\[\square\]

In Hamiltonian mechanics, the primary purpose of finding symmetries is to determine conservation laws. These two concepts are linked through Noether’s theorem. We have seen, in Theorem 3.1, that in the present context if a functional \(E\) satisfies the permutation symmetry (3.4), then the functional
gives rise to a Hamiltonian \( \mathcal{H} \) and Hamilton’s equation of motion is \( iu_t = \mathcal{T}(u, \ldots, u) \). With this Hamiltonian structure, a version of Noether’s Theorem applies.

**Theorem 3.5** (Noether’s Theorem). Let \( \mathcal{E} \) be a multilinear functional that satisfies the permutation symmetry (3.4). Suppose that \( Q \) is a self-adjoint operator on \( L^2 \) such that

\[
\mathcal{E}(e^{i\lambda Q} f_1, \ldots, e^{i\lambda Q} f_{2n}) = \mathcal{E}(f_1, \ldots, f_{2n}).
\]

Then the quantity \( \langle Qf, f \rangle_{L^2} \) is conserved by the Hamiltonian flow of \( \mathcal{H}(f) = \mathcal{E}(f, \ldots, f) \).

**Proof.** We first show that \( \langle \mathcal{T}(f, \ldots, f), Qf \rangle \in \mathbb{R} \). Differentiating equation (3.15) with respect to \( \lambda \) and setting \( \lambda = 0 \) gives,

\[
\sum_{k=1}^{n} \mathcal{E}(f, \ldots, f, Qf, f, \ldots, f) = \sum_{k=1}^{n} \mathcal{E}(f, \ldots, f, Qf, f, \ldots, f);
\]

the sign being determined by the linearity or conjugate linearity of each component. Now using the permutation symmetry (3.4) followed by (3.16) gives,

\[
\langle \mathcal{T}(f, \ldots, f), Qf \rangle = 2 \sum_{k=1}^{n} \mathcal{E}(f, \ldots, f, Qf, f, \ldots, f) = 2 \sum_{k=1}^{n} \mathcal{E}(f, \ldots, f, Qf, f, \ldots, f) = \langle \mathcal{T}(f, \ldots, f), Qf \rangle,
\]

which shows that \( \langle \mathcal{T}(f, \ldots, f), Qf \rangle \) is real. Then, if \( i\mathcal{T} = \mathcal{T}(f, \ldots, f) \), we have

\[
\frac{d}{dt} \langle Qf, f \rangle = i \left[ \langle Q\mathcal{T}(f, \ldots, f), f \rangle - \langle Qf, \mathcal{T}(f, \ldots, f) \rangle \right]
\]

\[
= i \left[ \langle \mathcal{T}(f, \ldots, f), Qf \rangle - \langle Qf, \mathcal{T}(f, \ldots, f) \rangle \right] = 2i \text{Im} \langle \mathcal{T}(f, \ldots, f), Qf \rangle = 0,
\]

so \( \langle Qf, f \rangle \) is constant.

The following table summarizes the relationships between the symmetries of \( E_A \) described in Theorem 3.3, the associated commuting operators in Corollary 3.4, and the conserved quantities given by Noether’s Theorem. As discussed previously, the functional \( E_A \) does not automatically satisfy the permutation symmetry (3.4) so Noether’s Theorem does not apply directly; however, the Hamiltonian systems defined by \( \mathcal{H}_6 \) and \( \mathcal{H}_4 \) do satisfy (3.4) and so will have a number of these conserved quantities as a consequence of symmetries induced by Theorem 3.3.

<table>
<thead>
<tr>
<th>Symmetry ( e^{i\lambda Q} ) of ( \mathcal{E} )</th>
<th>Operator ( Q ) commuting with ( \mathcal{T} )</th>
<th>Conserved quantity ( \langle Qf, f \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f \mapsto e^{i\lambda f} )</td>
<td>( 1 )</td>
<td>( \int_{\mathbb{R}}</td>
</tr>
<tr>
<td>( f \mapsto f_\lambda )</td>
<td>( x^2 )</td>
<td>( \int_{\mathbb{R}}</td>
</tr>
<tr>
<td>( f \mapsto e^{i\lambda</td>
<td>x</td>
<td>^2 f} )</td>
</tr>
<tr>
<td>( f \mapsto e^{i\lambda H} f )</td>
<td>( H )</td>
<td>( \int_{\mathbb{R}}</td>
</tr>
<tr>
<td>( f \mapsto e^{i\lambda x} f )</td>
<td>( x )</td>
<td>( \int_{\mathbb{R}} x</td>
</tr>
<tr>
<td>( f \mapsto f(\cdot + \lambda) )</td>
<td>( \frac{id}{dx} )</td>
<td>( \int_{\mathbb{R}} if'(x)\bar{f}(x) dx )</td>
</tr>
</tbody>
</table>

### 3.3. Boundedness and wellposedness

In this subsection we establish an \( L^2 \) bound on \( E_A \), and bounds of the form,

\[
\|T_A(f_1, \ldots, f_{2n-1})\|_X \leq C\|f_1\|_X \cdots \|f_{2n-1}\|_X,
\]

for \( X = L^2 \), \( X = L^{2,\sigma} \) and \( X = H^\sigma \). We will then show how these bounds imply local existence of solutions to Hamilton’s equation \( iu_t = T_A(u, \ldots, u) \) in the space \( X \). By employing some of the
conservation laws derived in the last section, it is possible to establish global existence in $L^2$ and other spaces for certain functionals.

**Theorem 3.6.** There holds the bound,

$$
|E_A(f_1, \ldots, f_{2n})| \leq \prod_{k=1}^{2n} \|f_k\|_{L^2}.
$$

In particular we have $H_A(f) \leq \|f\|_{L^2}^2$. Both of these bounds are sharp.

**Proof.** The proof involves one use of the Cauchy–Schwarz inequality. We have,

$$
|E_A(f_1, \ldots, f_{2n})| = \left| \int \prod_{k=1}^{n} f_k((Ax)_k)^2 \right| \leq \left( \int \prod_{k=1}^{n} |f_k((Ax)_k)|^2 \right)^{1/2} \left( \int \prod_{k=1}^{n} |f_{n+k}(x_k)|^2 \right)^{1/2}.
$$

In the first integral we perform the change of variables $y = Ax$. Because $A$ is an isometry, the determinant of this change of variables is 1, and so by Fubini’s Theorem,

$$
|E_A(f_1, \ldots, f_{2n})| \leq \left( \int \prod_{k=1}^{n} |f_k((Ax)_k)|^2 \right)^{1/2} \left( \int \prod_{k=1}^{n} |f_{n+k}(x_k)|^2 \right)^{1/2} = \prod_{k=1}^{2n} \|f_k\|_{L^2},
$$

which is the bound for $E_A$. Setting $f_k = f$ for all $k$ gives the bound $H_A(f) \leq \|f\|_{L^2}^2$.

From the Cauchy–Schwarz inequality, we have the equality condition,

$$
\prod_{k=1}^{n} f_k((Ax)_k) = \prod_{k=1}^{n} f_{n+k}(x_k).
$$

By assumption, $A$ is an isometry, so that $\sum_{k=1}^{n} |(Ax)_k|^2 = |Ax|^2 = |x|^2 = \sum_{k=1}^{n} |x_k|^2$. This immediately gives that the Gaussian $G(x) = e^{-\alpha x^2}$ satisfies the equality condition (3.19), and hence that $H(G) = \|G\|_{L^2}^{2n}$. The inequality (3.18) is thus sharp. \[\square\]

In a later subsection we discuss the classification of all functions $\{f_k\}_{k=1}^{2n}$ that saturate the multilinear functional inequality (3.18). The two examples on page 14 show that, in general, such a classification will depend on the matrix properties of $A$. In Example 3.1, $H_A(f) = \|f\|_{L^2}^2$, and so the bound $H_A(f) \leq \|f\|_{L^2}^2$ in Theorem 3.6 is always equality. This is not the case in Example 3.2. In Theorem 3.11, we will find that if $A$ is not a signed permutation matrix – namely that there is at least one basis element $e_k$ such that $Ae_k$ is a linear sum of at least two other $e_j$ basis elements – then the equality $H_A(f) = \|f\|_{L^2}^2$ holds only if $f$ is a Gaussian.

In the meantime, we use the inequality (3.18) to establish bounds for the multilinear operator $T_A$.

**Corollary 3.7.** There holds the bound,

$$
\|T_A(f_1, \ldots, f_{2n-1})\|_X \leq C_n \prod_{k=1}^{2n-1} \|f_k\|_X^{2n-1}
$$

for the following spaces.

(i) $X = L^2$ with $C_n = 2n$;

(ii) $X = L^{2,\sigma}$, for any $\sigma \geq 0$ with $C_n = 2n^{2+\sigma}$.

(iii) $X = H^\sigma$, for any $\sigma \geq 0$ with $C_n = 2n^{2+\sigma}$.

It is important that the boundedness constant $C_n$ is independent of $A$. This implies that if we have a composite Hamiltonian of the form,

$$
\int_{\Omega} \phi(\lambda) H_A(\lambda)(f) d\lambda,
$$

then the associated operator will be bounded once $\phi$ is integrable. This is precisely how Corollary 3.7 will be applied to the Hamiltonian systems $\mathcal{H}_6$ and $\mathcal{H}_4$. 
Proof. (i) We argue by duality using the implicit representation (3.3). By the bound on \( E_A \) from Theorem 3.6 we have

\[
|\langle T_A(f_1, \ldots, f_{2n-1}), g \rangle| \leq 2 \sum_{k=n+1}^{2n} |E_A(f_1, \ldots, f_{n+m-1}, g, f_{n+m}, \ldots, f_{2n-1})|
\]

\[
\leq 2 \sum_{k=n+1}^{2n} 2^{n-1} \prod_{k=1}^{2n-1} \|f_k\|_{L^2} \|g\|_{L^2} = \left(2n \prod_{k=1}^{2n-1} \|f_k\|_{L^2}\right) \|g\|_{L^2}
\]

This gives the result for \( X = L^2 \).

(ii) Fix \( x \in \mathbb{R}^n \). Because \( A \) is an isometry we have, for every \( m \), \(|x_m|^2 \leq |x|^2 = |Ax|^2 = \sum_{k=1}^n |(Ax)_k|^2\). Therefore, for fixed \( m \), there is an integer \( l \) such that \(|x_m|^2 \leq n|(Ax)_l|^2\). We then have \(|x_m| \leq n|(Ax)_l| \) and so,

\[
\langle x_m \rangle \leq n \left( \prod_{k=1, k \neq m}^{n} \langle (Ax)_k \rangle \langle x_k \rangle \right) \langle (Ax)_m \rangle,
\]

because in all cases \(|t| \geq 1 \). In terms of the functional \( E_A \), this gives,

\[
E_A(|f_1|, \ldots, |f_{k-1}|, \langle t \rangle^\sigma |f_k|, |f_{k-1}|, \ldots, |f_{2n}|) \leq n^\sigma E_A(|t \rangle^\sigma |f_1|, \ldots, \langle t \rangle^\sigma |f_{k-1}|, |f_k|, \langle t \rangle^\sigma |f_{k-1}|, \ldots, \langle t \rangle^\sigma |f_{2n}|).
\]

Now applying this to \( T_A \), we have,

\[
\langle T_A(f_1, \ldots, f_{2n-1}), g \rangle_{L^2, \sigma} = \langle T_A(f_1, \ldots, f_{2n-1}), \langle t \rangle^{2\sigma} g \rangle_{L^2, \sigma} = 2n \sum_{k=n+1}^{2n} E_A(f_1, \ldots, f_{k-1}, \langle t \rangle^{2\sigma} g, f_k, \ldots, f_{2n-1}) \leq 2n^{1+\sigma} \sum_{k=n+1}^{2n} E_A(|f_1|, \ldots, \langle t \rangle^\sigma |f_{k-1}|, |f_k|, \langle t \rangle^\sigma |f_{k-1}|, \ldots, \langle t \rangle^\sigma |f_{2n-1}|) \leq \left(2n^{2+\sigma} \prod_{k=1}^{2n-1} \|f_k\|_{L^2, \sigma} \right) \|g\|_{L^2, \sigma},
\]

which gives the result for \( X = L^{2, \sigma} \).

(iii) This follows from (ii) using the invariance of the operator \( T_A \) under the Fourier transform. \( \square \)

The next proposition shows that, in general, bounds of the form determined in the previous proposition imply local wellposedness.

**Proposition 3.8.** Suppose that \( T : X^m \to X \) is a multilinear operator that is bounded on a complete normed vector space \( X \); that is,

\[
\|T(f_1, \ldots, f_m)\|_X \leq C_T \prod_{k=1}^{m} \|f_k\|_X
\]

for some \( C_T \). Set \( T(f) = T(f, \ldots, f) \). For every \( f_0 \in X \) there is a \( T > 0 \) and unique local solution to the Cauchy problem for Hamiltonian’s equation,

\[
if_t = T(f),
\]

\[
f(t = 0) = f_0,
\]

in the space \( L^\infty([0, T], X) \). The time of existence \( T \) depends only on \( \|f_0\|_X \).

**Proof.** The Duhamel formulation of the Cauchy problem (3.21) is

\[
f(t) := R(f(t)) = f_0 + \int_{0}^{t} T(f(s))ds.
\]
To prove local existence we will show that there is a $T = T(\|f_0\|_X)$ such that the operator $R$ is a contraction on $Y_T = L^\infty([0,T],X)$. The proposition then follows from Banach’s Fixed Point Theorem.

We first have the bound
\begin{equation}
R(f(t)) \leq f_0 + \int_0^t \|T(s)\|_X ds \leq f_0 + C_T \int_0^t \|f(s)\|_X ds \leq f_0 + tC_T \|f\|_Y^m.
\end{equation}

Next we have $\|R(f_1(t)) - R(f_2(t))\| \leq f_0^t \|T(f_1(s)) - T(f_2(s))\|_X ds$. To evaluate the right-hand side, we use multilinearity to write,
\[ T(f_1(s)) - T(f_2(s)) = T(f_1, \ldots, f_1) - T(f_2, \ldots, f_2) = \sum_{k=1}^m T(f_1, \ldots, f_1, f_1 - f_2, f_2, \ldots, f_2), \]
This equality gives,
\[ \|T(f_1(s)) - T(f_2(s))\| \leq C_T \sum_{k=1}^{m-1} \|f_1\|_X^{k-1} \|f_1 - f_2\|_X \|f_2\|_X^{m-k} \]
\[ \leq \|f_1 - f_2\|_X (m-1)C_T \left[ \|f_1\|_Y^{m-1} + \|f_2\|_Y^{m-1} \right], \]
and then,
\begin{equation}
R(f_1(t)) - R(f_2(t)) \leq \|f_1 - f_2\|_Y (t(m-1)C_T \left[ \|f_1\|_Y^{m-2} + \|f_2\|_Y^{m-2} \right]).
\end{equation}

From (3.22) and (3.23) it follows that for any $M > \|f_0\|_X$ we may choose $T$ such that $R : B_{Y_T}(0,M) \to B_{Y_T}(0,M)$ and $R$ a contraction on this space.

It is key in the proposition that the local time of existence depends only on $\|f_0\|_X$. This enables one to pair the local wellposedness result with a conservation law in the previous subsection to get global wellposedness. We will do precisely this to prove global wellposedness in $L^2$ of solutions to the Hamiltonian systems $\mathcal{H}_{6}$ and $\mathcal{H}_{4}$.

3.4. The functional in the basis of Hermite functions. The symmetry of the functional under the action $e^{itH}$ implies interesting explicit relationships between $E_A$, $T_A$ and the Hermite functions $\{\phi_n\}_{n=0}^\infty$.

**Theorem 3.9.** If $\sum_{k=1}^n m_k - \sum_{k=n+1}^{2n} m_k \neq 0$, then $E_A(\phi_{m_1}, \ldots, \phi_{m_{2n}}) = 0$. This implies that
\begin{equation}
T_A(\phi_{m_1}, \ldots, \phi_{m_{2n-1}}) = C \phi_l,
\end{equation}
for some constant $C$ where $l = \sum_{k=1}^n m_k - \sum_{k=n+1}^{2n-1} m_k$. (We understand that if $l < 0$ then $\phi_l := 0$.)

**Proof.** By the symmetry under the action $e^{itH}$, we have
\[ E_A(\phi_{m_1}, \ldots, \phi_{m_{2n}}) = E_A(e^{itH} \phi_{m_1}, \ldots, e^{itH} \phi_{m_{2n}}) \]
\[ = E_A(e^{it(2m_1+1)} \phi_{m_1}, \ldots, e^{it(2m_{2n-1}+1)} \phi_{m_{2n}}) \]
\[ = e^{it2L} E_A(\phi_{m_1}, \ldots, \phi_{m_{2n}}) \]
where $L = \sum_{k=1}^n m_k - \sum_{k=n+1}^{2n} m_k$. If $L \neq 0$, as in the statement of the theorem, then necessarily the $E_A$ term here is 0.

Next, let $m_1, \ldots, m_{2n-1}$ be given, and fix $p \neq \sum_{k=1}^n m_k - \sum_{k=n+1}^{2n-1} m_k$. We have,
\[ (T_A(\phi_{m_1}, \ldots, \phi_{m_{2n-1}}), \phi_p)_{L^2} = 2 \sum_{k=1}^n E_A(\phi_{m_1}, \ldots, \phi_{m_{n+k-1}}, \phi_p, \phi_{m_{n+k}}, \ldots, \phi_{m_{2n}}) = 0, \]
by the previous part and because $p \neq \sum_{k=1}^n m_k - \sum_{k=n+1}^{2n-1} m_k$. The function $T_A(\phi_{m_1}, \ldots, \phi_{m_{2n-1}})$ is thus orthogonal to $\phi_p$ for all such $p$. Because the Hermite functions are an orthonormal basis of $L^2$, we must have (3.24).

**Corollary 3.10.** Set $\omega_n = 2nH_A(\phi_n)$. Then $u(t,x) = e^{it\omega_n} \phi_n(x)$ is a stationary wave solution of $iu_t = T_A(u, \ldots, u)$. 
Proof. From the previous proposition, we know that,

\[ T_A(\phi_n, \ldots, \phi_n) = \omega_n \phi_n, \]

for some \( \omega_n \). It follows that \( e^{-i\omega_n \phi_n(x)} \) is a solution of \( iu_t = T_A(u, \ldots, u) \).

Taking the inner product of both sides of (3.25) with \( \phi_n \) and using (3.3), we find that
\[
\omega_n = 2n H_A(\phi_n).
\]

By applying the symmetries of \( E_A \) given in Theorem 3.3 to \( \phi_n \), one can obtain other stationary wave solutions. In fact, with this procedure one finds that all functions of the form, \( ae^{ibx^2} \phi_n(cx) \) are stationary waves, where \( a \in \mathbb{C} \) and \( b, c \in \mathbb{R} \). If in addition the matrix \( A \) satisfies \( A(1, \ldots, 1) = (1, \ldots, 1) \), then all functions of the form, \( ae^{ibx^2 + idx} \phi_n(cx + e) \) are stationary waves, where \( a \in \mathbb{C} \) and \( b, c, d, e \in \mathbb{R} \).

We note that in higher dimensions, as in the work of [17], the dynamics of the eigenvectors of \( H \) under the continuous resonant equation are more interesting because the eigenspaces of \( H \) are multidimensional. This allows for more complicated dynamics than simply stationary waves. In dimension one, each eigenspace of \( H \) is one dimensional (being spanned by \( \phi_n \) alone) and it necessarily follows that any eigenvector of \( H \) corresponds simply to a stationary wave solution.

### 3.5. Classification of the maximizers of the \( L^2 \) bound

We have previously established the inequality \( |H_A(f)| \leq \|f\|_{L^2}^2 \) and showed that it is sharp. In this section we discuss the classification those functions that saturate the inequality. While classifying the cases of equality in an inequality is often illuminating, in the theory of Hamiltonians on the phase space \( L^2 \) it is especially relevant. In all the Hamiltonian systems discussed in this article, the \( L^2 \) norm of \( f \) is conserved by the Hamiltonian flow, and the Hamiltonian \( H \) itself is always conserved. Therefore, if one takes as initial data to the flow a function that saturates the inequality \( H(f) \leq \|f\|_{L^2}^2 \), for all fixed times \( t \) the solution \( x \mapsto f(x, t) \) will still saturate the inequality. The set of all saturating functions is thus closed under the flow. Identifying the set of saturating functions can then lead to interesting explicit solutions. For the Hamiltonians in this article, however, we will find that the saturating functions all correspond simply to stationary wave solutions.

The two examples on 14 show that the set of saturating functions for the inequality \( |H_A(f)| \leq \|f\|_{L^2}^2 \) will depend on the matrix properties of \( A \). Example 3.1 shows that if \( A \) is the identity, or more generally a permutation, then the inequality is always equality. If \( A \) is a signed permutation, for example the negative of the identity, then there is equality if and only if \( f \) is even, as we will see. However if \( A \) is not a signed permutation, as in Example 3.2, then the set of saturating functions is much smaller.

**Theorem 3.11.** Suppose that \( A \) is an isometry which is not a signed permutation – that is, there is some \( i \) such that \( Ae_i \) is the linear sum of at least two basis elements \( e_j \). Let \( e = (1, \ldots, 1) \) denote the vector in \( \mathbb{R}^n \) with all its entries 1.

(i) If \( Ae = e \), then we have the equality \( H_A(f) = \|f\|_{L^2}^2 \) if and only if \( f(x) = ce^{-\alpha x^2 + \beta x} \) for some \( c, \alpha, \beta \in \mathbb{C} \) with \( \text{Re} \alpha > 0 \).

(ii) If \( Ae \neq e \), then we have the equality \( H_A(f) = \|f\|_{L^2}^2 \) if and only if \( f(x) = ce^{-\alpha x^2} \) for some \( c, \alpha \in \mathbb{C} \) with \( \text{Re} \alpha > 0 \).

**Proof of the ‘if’ statements.** To check the equality case for Gaussians \( f(x) = ce^{-\alpha x^2 + \beta x} \), we substitute \( f \) into both sides of (3.19), which was the equality condition in our use of the Cauchy–Schwarz inequality. The left hand side is,

\[
(3.26) \prod_{k=1}^n f(A(x)_k) = c^n \prod_{k=1}^n \exp \left( -\alpha \langle Ax, e_k \rangle^2 + \beta \langle x, e_k \rangle \right) = c^n \exp \left( -\alpha \sum_{k=1}^n \langle Ax, e_k \rangle^2 + \beta \langle Ax, e_k \rangle \right)
\]

\[
(3.27) = c^n \exp \left( -\alpha |Ax|^2 + \beta \langle x, e \rangle \right) = c^n \exp \left( -\alpha |Ax|^2 + \beta \langle x, A^{-1}e \rangle \right),
\]

where in the last line we have used \( \langle Ax, e \rangle = \langle x, A^{-1}e \rangle \) coming from \( A \) being an isometry.

An identical computation shows that the right hand side of (3.19) is,

\[
\prod_{k=1}^n f(x_k) = c^n \exp \left( -\alpha |x|^2 + \beta \langle x, v \rangle \right).
\]
If \( Ae = e \), so that \( e = A^{-1}e \), then both sides are equal for all \( c, \alpha \) and \( \beta \), which proves the ‘if’ part of item (i). On the other hand, if \( Ae \neq e \) then there is an \( x \in \mathbb{R}^n \) such that \( (x, A^{-1}e) \neq (x, e) \). Hence for equality to hold we necessarily have \( \beta = 0 \). However with \( \beta = 0 \) both sides are equal, and so the ‘if’ part of item (ii) is proved. \( \square \)

The ‘only if’ part of the theorem follows from a more general result that classifies those functions that saturate the multilinear functional inequality \( |E_A(f_1, \ldots, f_{2n})| \leq \|f_1\|_{L^2} \cdots \|f_{2n}\|_{L^2} \). Our classification result, given in Theorem 3.12 below, is not new. The multilinear functional inequality (3.18) is a specific example of a geometric Brascamp–Lieb inequality. The Brascamp–Lieb inequalities originated in [6] as generalizations of the Hölder inequality. The special class of geometric Brascamp–Lieb inequalities was introduced in [1], and it was subsequently shown, in [2], that they are maximized only for Gaussians. Theorem 3.12 is a special case of this broad result. Our proof of this special case is, it seems, new, and we think worthy of presentation; however, because the result is not new and not essential to other parts of this article, the proof is presented in Appendix A.

**Theorem 3.12.** Let \( A \) be an isometry. Denote \( f_k(x) = f_k(-x) \). There exists integers \( m \) and \( l \), with \( 0 \leq m \leq l \leq n \), and two permutations \( \sigma_1 \) and \( \sigma_2 \) of the integers \( \{1, \ldots, n\} \) such that

\[
E_A(f_1, \ldots, f_{2n}) = \prod_{k=1}^{m} \langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)} \rangle \prod_{k=m+1}^{l} \langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)} \rangle E_B(f_{\sigma_1(l+1)}, \ldots, f_{\sigma_1(n)}, f_{n+\sigma_2(l+1)}, \ldots, f_{n+\sigma_2(n)}),
\]

where the matrix \( B : \mathbb{R}^{n-l} \to \mathbb{R}^{n-l} \) has no permutation part; that is, for all \( k \) and \( j \), \( B e_k \neq \pm e_j \).

We then have \( |E_A(f_1, \ldots, f_{2n})| = \|f_1\|_{L^2} \cdots \|f_{2n}\|_{L^2} \) if and only if the following three conditions hold.

1. For \( k = 1, \ldots, m \), \( f_{\sigma_1(k)}(x) = C_k f_{n+\sigma_2(k)}(x) \), for some \( C_k \in \mathbb{R} \).
2. For \( k = m+1, \ldots, l \), \( f_{\sigma_1(k)}(x) = C_k f_{n+\sigma_2(k)}(-x) \), for some \( C_k \in \mathbb{R} \).
3. For \( k = 1, \ldots, m \), \( f_{\sigma_1(k)}(x) \) and \( f_{n+\sigma_2(k)}(x) \) are Gaussians.

Proof of the ‘only if’ statements in Theorem 3.11, assuming Theorem 3.12. Because \( A \) is not a signed permutation, we must have \( l < n \) in the notation of Theorem 3.12. Then in, Theorem 3.12, condition 3 applies. Hence to have the equality \( H_A(f) = E_A(f, \ldots, f) = \|f\|_{L^2}^{2n} \), \( f \) must necessarily be a Gaussian. \( \square \)

### 3.6. Regularity of stationary waves: a lemma.

In this final part of our general study of the functional \( E_A \), we prove a weight transfer lemma that is similar to formula (3.20). This lemma will be crucial later to proving that stationary waves of the Hamiltonian systems (1.7) and (1.9) equation are analytic and exponentially decaying once they are in \( L^2 \).

For fixed \( \mu, \epsilon > 0 \), define,

\[
G_{\mu,\epsilon}(x) = \exp \left( \frac{\mu x^2}{1 + \epsilon x^2} \right).
\]

**Lemma 3.13.** If \( \{f_k\}_{k=1}^{2n} \) are positive functions, then

\[
E_A(f_1, \ldots, f_{2n-1}, f_{2n} G_{\mu,\epsilon}) \leq E_A(f_1 G_{\mu,\epsilon}, \ldots, f_{2n-1} G_{\mu,\epsilon}, f_{2n}).
\]

**Proof.** Define \( F_{\mu,\epsilon} = \mu |x|/(1 + \epsilon |x|) \). We clearly have \( G_{\mu,\epsilon}(x) = \exp(F_{\mu,\epsilon}(x^2)) \).

We record two properties of \( F_{\mu,\epsilon} \). First, for \( x > 0 \), \( F_{\mu,\epsilon} \) is increasing. This may be seen from,

\[
\frac{d}{dx} \left( \frac{x}{1 + \epsilon x} \right) = \frac{1 + \epsilon x - x(\epsilon)}{(1 + \epsilon x)^2} = \frac{\epsilon x}{(1 + \epsilon x)^2} > 0.
\]

Next, we have \( F_{\mu,\epsilon}(x_1 + x_2) \leq F_{\mu,\epsilon}(x_1) + F_{\mu,\epsilon}(x_2) \). This may be seen from,

\[
F_{\mu,\epsilon}(x_1 + x_2) = F_{\mu,\epsilon}(|x_1 + x_2|) \leq F_{\mu,\epsilon}(|x_1| + |x_2|) = \mu \frac{|x_1| + |x_2|}{1 + \epsilon |x_1| + \epsilon |x_2|} \leq \mu \frac{|x_1|}{1 + \epsilon |x_1|} + \mu \frac{|x_2|}{1 + \epsilon |x_2|} = F_{\mu,\epsilon}(x_1) + F_{\mu,\epsilon}(x_2).
\]
Now because $A$ is an isometry we have, for all $x \in \mathbb{R}^n$, $x_n^2 = \sum_{k=1}^n (Ax_k)^2 - \sum_{k=1}^{n-1} (x_k)^2$ and hence by the sublinearity property of $F_{\mu,\epsilon},$

$$F_{\mu,\epsilon}(x_n^2) = F_{\mu,\epsilon} \left( \sum_{k=1}^n (Ax_k)^2 + \sum_{k=1}^{n-1} (x_k)^2 \right) \leq \sum_{k=1}^n F_{\mu,\epsilon}((Ax_k)^2) + \sum_{k=1}^{n-1} F_{\mu,\epsilon}(x_k^2),$$

Then, because $x \mapsto e^x$ is increasing,

$$G_{\mu,\epsilon}(x_n) = \exp(F_{\mu,\epsilon}(x_n^2)) \leq \prod_{k=1}^n G_{\mu,\epsilon}((Ax_k)k) \prod_{k=1}^{n-1} G_{\mu,\epsilon}(x_k).$$

Applying this to $E_A$, we have,

$$E_A(f_1, \ldots, f_{2n-1}, f_{2n}G_{\mu,\epsilon}) = \int_{\mathbb{R}^n} \left( \prod_{k=1}^{n-1} f_k((Ax_k)\), \bar{f}_n((Ax_n)\) \right) f_n((Ax_n)\) \bar{f}_n((Ax_n)\) G_{\mu,\epsilon}(x_n) dx$$

$$\leq \int_{\mathbb{R}^n} \left( \prod_{k=1}^{n-1} f_k((Ax_k)\)G_{\mu,\epsilon}((Ax_k)\), \bar{f}_n((Ax_n)\)G_{\mu,\epsilon}(x_n) \right) f_n((Ax_n)\)G_{\mu,\epsilon}((Ax_n)\) \bar{f}_n((Ax_n)\) G_{\mu,\epsilon}(x_n) dx$$

$$= E_A(f_1G_{\mu,\epsilon}, \ldots, f_{2n-1}G_{\mu,\epsilon}, f_{2n}),$$

which is what we wanted to prove. \qed

4. The Quintic Resonant Equation

We now turn to the first resonant Hamiltonian,

$$\mathcal{H}_6(f) = \frac{2}{\pi} \|e^{it\mathcal{H}} f\|_{L_6^6} = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} |e^{it\mathcal{H}} f(x)|^6 dx dt,$$

which has a corresponding multilinear functional,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} (e^{it\mathcal{H}} f_1)(e^{it\mathcal{H}} f_2)(e^{it\mathcal{H}} f_3)(e^{it\mathcal{H}} f_4)(e^{it\mathcal{H}} f_5)(e^{it\mathcal{H}} f_6) dx dt.$$

We point out right away that, in contrast to the multilinear functionals $E_A$ studied in the previous section, the functional $\mathcal{E}_6$ has a large number of permutation symmetries. For any two permutations of three elements $\sigma, \sigma' \in S_3$, we have,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \mathcal{E}_6(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}, f_{\sigma'(4)}, f_{\sigma'(5)}, f_{\sigma'(6)})$$

as well as the standard symmetry,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \mathcal{E}_6(f_4, f_5, f_6, f_1, f_2, f_3).$$

These symmetries simplify the formulas for Hamilton’s equation. In fact, by Theorem 3.1, $\mathcal{H}_6$ defines a Hamiltonian flow on the phase space $L^2$ with Hamilton’s equation given by

$$iu_t = \mathcal{T}_6(u, u, u, u),$$

where $\mathcal{T}_6$ is defined by the simple relation,

$$\langle \mathcal{T}_6(f_1, f_2, f_3, f_4, f_5, g) \rangle_{L^2} = 6\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, g).$$

Theorem 4.1. There holds the representation,

$$\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5, f_6)(x) = \frac{12}{\pi} \int_{-\pi/4}^{\pi/4} e^{-it\mathcal{H}} \left[ (e^{it\mathcal{H}} f_1)(e^{it\mathcal{H}} f_2)(e^{it\mathcal{H}} f_3)(e^{it\mathcal{H}} f_4)(e^{it\mathcal{H}} f_5)(e^{it\mathcal{H}} f_6) \right](x) dt.$$
Proof. Using (4.6) we have,
\[
\langle T_0(f_1, f_2, f_3, f_4, f_5, g) \rangle_{L^2} = 6\mathcal{E}(f_1, f_2, f_3, f_4, f_5, g)
\]
\[
= \frac{12}{\pi} \int_0^{\pi/2} \left\{ (e^{itH} f_1)(e^{itH} f_2)(e^{itH} f_3)(e^{itH} f_4)(e^{itH} f_5), e^{itH} g) \right\} L^2(\mathbb{R}) dt
\]
\[
= \frac{12}{\pi} \int_0^{\pi/2} \left\{ e^{-itH} \left[ (e^{itH} f_1)(e^{itH} f_2)(e^{itH} f_3)(e^{itH} f_4)(e^{itH} f_5), g \right] \right\} L^2(\mathbb{R}) dt,
\]
where we have used the fact that $e^{itH}$ is an isometry of $L^2$ with inverse $e^{-itH}$. Upon commuting the space and time integrations, we get (4.7). □

This theorem shows that the Hamiltonian flow corresponding to $\mathcal{H}_6$ is precisely the resonant equation 2.9 in the quintic case $k = 2$. By the approximation result, Theorem 2.3, solutions of (4.5) with initial data of size $\epsilon$ are close to solutions of $i u_t - \Delta u + |x|^2 = |u|^4 u$ in the space $\mathcal{H}^s$ for $s > 1/2$ and times $t \lesssim \epsilon^{-5}$.

4.1. Representations of the Hamiltonian and the flow operator. We discussed in the introduction that a significantly useful approach to the study of Hamiltonians such as $\mathcal{H}_6$ is to determine alternative representation formulas for $H_6$, $\mathcal{E}_6$, and $T_0$. Functionals such as $\mathcal{E}_6$ can have a large amount of structure that is concealed by a specific representations such as (4.2). This is will be illustrated clearly below.

First, we show that $\mathcal{E}_6$ is invariant under the Fourier transform. In fact, this invariance will be a simple consequence of Theorem 3.2 (which showed Fourier transform invariance for $E_A$) once we have established a specific representation of $\mathcal{E}_6$ in Theorem 4.7 below. However we wish to employ Fourier transform invariance before establishing that representation.

Lemma 4.2. The functional $\mathcal{E}_6$ and operator $T_0$ are invariant under the Fourier transform,
\[
\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \mathcal{E}_6(\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5, \hat{f}_6),
\]
\[
T_0(f_1, f_2, f_3, f_4, f_5, f_6) = T_0(\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5, \hat{f}_6).
\]
Proof. First let $f_k = \phi_{n_k}$ be Hermite functions. Then $\hat{\phi}_{n_k} = (i)^{n_k} \phi_{n_k}$, and so,
\[
\mathcal{E}_6(\phi_{n_1}, \phi_{n_2}, \phi_{n_3}, \phi_{n_4}, \phi_{n_5}, \phi_{n_6}) = (i)^{n_1+n_2+n_3-n_4-n_5-n_6} \mathcal{E}_6(\phi_{n_1}, \phi_{n_2}, \phi_{n_3}, \phi_{n_4}, \phi_{n_5}, \phi_{n_6}).
\]
On the other hand,
\[
\mathcal{E}_6(\phi_{n_1}, \ldots, \phi_{n_6}) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} e^{2i(n_1+n_2+n_3-n_4-n_5-n_6)} dt
\]
\[
\times \int_{\mathbb{R}} \phi_{n_1}(x) \phi_{n_2}(x) \phi_{n_3}(x) \phi_{n_4}(x) \phi_{n_5}(x) \phi_{n_6}(x) dx
\]
If $n_1 + n_2 + n_3 - n_4 - n_5 - n_6$ is a nonzero even integer, then the time integral in (4.11) is 0. If $n_1 + n_2 + n_3 - n_4 - n_5 - n_6$ is an odd integer, then by the property $\phi_{n_k}(-x) = (-1)^{n_k} \phi_{n_k}(x)$, the integrand in the space integral in (4.11) is odd and hence the integral is 0. Therefore, using also (4.10), if $n_1 + n_2 + n_3 - n_4 - n_5 - n_6 \neq 0$, both $\mathcal{E}(\phi_{n_1}, \ldots, \phi_{n_6})$ and $\mathcal{E}(\phi_{n_1}, \ldots, \phi_{n_6})$ are 0 and in particular equal. Moreover, if $n_1 + n_2 + n_3 - n_4 - n_5 - n_6 = 0$, then by (4.10) $\mathcal{E}(\phi_{n_1}, \ldots, \phi_{n_6}) = \mathcal{E}(\phi_{n_1}, \ldots, \phi_{n_6})$.

Because the Hermite functions are a basis of $L^2$, the formula (4.8) holds for all functions $f_k$. The operator statement then follows from the same computation (3.11) as in the proof of Theorem 3.2. □

Theorem 4.3. There holds the representations,
\[
\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5)(e^{it\Delta} f_6) dx dt,
\]
\[
T_0(f_1, f_2, f_3, f_4, f_5)(x) = \frac{12}{\pi} \int_{-\pi/4}^{\pi/4} e^{-it\Delta} \left[ (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5) \right] (x) dt.
\]
Proof. Recall that the lens transform (3.12) takes solutions $u$ of the linear Schrödinger equation into solutions $v$ of the linear Schrödinger equation with harmonic trapping. If we let $u_k(x, t) = (e^{itH} f_k)(x)$ and $v_k(x) = (e^{it\Delta} f_k)(x)$, the lens transform gives,

$$u_k(x, t) = \frac{1}{\cos^2(2t)^{1/2}} v_k \left( \frac{x}{\cos(2t)} - \tan(2t) \frac{\tan(2t)}{2} \right) e^{i\pi/2}. $$

We substitute these expressions into (4.2) and perform two changes of variable. In the time variable, we perform $s = \frac{1}{2} \tan(2t)$. This change of variables bijectively maps $(-\pi/4, \pi/4)$ to $(-\infty, \infty)$ and has determinant $\cos(2t)^{-2}$. In the space variable we perform $y = x/\cos(t)$; this has determinant $|\cos(2t)|^{-1}$. Then,

$$\int_{-\pi/4}^{\pi/4} \int \left( u_1 u_2 u_3 \overline{u_4 u_5 u_6} \right)(x, t) dxdt = \int_{0}^{\pi/2} \frac{1}{\cos^2(2t)^{1/2}} \int \left( v_1 v_2 v_3 \overline{v_4 v_5 v_6} \right) \left( \frac{x}{\cos(2t)} - \tan(2t) \frac{\tan(2t)}{2} \right) (x, t) dxdt$$

$$= \int_{-\infty}^{\infty} \int \left( v_1 v_2 v_3 \overline{v_4 v_5 v_6} \right) (y, s) (x, t) dyds,$$

which gives (4.12). The expression for $\mathcal{T}_6$ follows from the same argument as in the proof of Theorem 4.1. \hfill \Box

Theorem 4.4. Let $\Omega_1(x) = y_1 + y_2 + y_3 - y_4 - y_5 - x$ and $\Omega_2(x) = y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 - x^2$. Then there holds the representations,

$$\mathcal{E}_0(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{\pi^2} \int_{\mathbb{R}} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) f_5(y_5) f_6(y_6) \delta_{\Omega_1(y_6)} \delta_{\Omega_2(y_6)} dy_6,$$

$$\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5, f_6)(x) = \frac{6}{\pi^2} \int_{\mathbb{R}} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) f_5(y_5) \delta_{\Omega_1(x)} \delta_{\Omega_2(x)} dy_6.$$

Proof. We evaluate (4.12) using the fundamental solution formula for the linear Schrödinger equation, $(e^{it\Delta} f_k)(x) = (4\pi it)^{-1/2} \int_{\mathbb{R}} e^{i|y|}\sqrt{i/4} f_k(y_6) dy_6$. Using, in addition, Fourier transform invariance of the functional and the Fourier inversion formula (1.18), we have,

$$\mathcal{E}_0(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi^2} \int_{\mathbb{R}} (e^{it\Delta} \hat{f}_1)(e^{it\Delta} \hat{f}_2)(e^{it\Delta} \hat{f}_3)(e^{it\Delta} \hat{f}_4)(e^{it\Delta} \hat{f}_5)(e^{it\Delta} \hat{f}_6) dxdt$$

$$= \frac{1}{32\pi^4} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\pi(y_1/2t + y_2/4t)} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) f_5(y_5) f_6(y_6) dy_1 dy_2 dy_3 dy_4 dy_5 dy_6,$$

$$= \frac{1}{4\pi^4} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\pi(y_1/2t + y_2/4t)} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) f_5(y_5) f_6(y_6) dy_1 dy_2 dy_3 dy_4 dy_5 dy_6$$

$$= \frac{2}{\pi^2} \int_{\mathbb{R}} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) f_5(y_5) f_6(y_6) \delta_{\Omega_1(y_6)} \delta_{\Omega_2(y_6)} dy_6,$$

which is (4.14). Equation (4.15) follows immediately from definition (4.6) with the $L^2$ inner product integration in $y_6$. \hfill \Box

Theorem 4.5. There holds the representations,

$$\mathcal{E}_0(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^2} \int_{\mathbb{R}} f_1(\beta + \xi) f_2(\lambda \beta + \gamma) f_3(\lambda \gamma + \xi - \lambda \xi)$$

$$\mathcal{T}_4(\lambda \beta + \xi) \mathcal{T}_5(\beta + \lambda \gamma + \xi - \lambda \xi) dy \frac{d\beta}{dy} d\xi d\gamma,$$

$$\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)(x) = \frac{3}{\pi^2} \int_{\mathbb{R}} f_1(\beta + \xi) f_2(\lambda \beta + x) f_3(\lambda x + \xi - \lambda \xi)$$

$$\mathcal{T}_4(\lambda \beta + \xi) \mathcal{T}_5(\beta + \lambda x + \xi - \lambda \xi) dy \frac{d\beta}{dy} d\xi.$$
Proof. We start with formula (4.17). Introduce new variables $\alpha, \beta, \gamma, \eta, \xi$ by $y_1 = \beta + \xi$, $y_2 = \eta + \gamma$, $y_3 = \alpha$, $y_4 = \eta + \xi$ and $y_5 = \alpha + \beta$. We calculate $y_6 = y_1 + y_2 + y_3 - y_4 - y_5 = \gamma$ and

$$
\Omega^2(y_6) = y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 = 2\beta\xi + 2\gamma\eta - 2\eta\xi - 2\alpha\beta,
$$

which gives the formula,

$$
E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^3} \int_{\mathbb{R}^6} e^{2it[\beta\xi + \gamma\eta - \eta\xi - \alpha\beta]}
$$

$$
f_1(\beta + \xi)f_2(\eta + \gamma)f_3(\alpha)\mathcal{T}_4(\eta + \xi)\mathcal{T}_5(\alpha + \beta)\mathcal{T}_6(\gamma)d\beta d\gamma d\eta d\xi dt
$$

Now change variables from $\eta$ to $\lambda$ through $\eta = \lambda\beta$. This gives $d\eta = |\beta|d\lambda$ and therefore,

$$
E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^3} \int_{\mathbb{R}^6} |\beta|e^{2it[\beta\xi + \gamma\lambda - \xi\lambda - \alpha]}
$$

$$
f_1(\beta + \xi)f_2(\lambda\beta + \gamma)f_3(\alpha)\mathcal{T}_4(\lambda\beta + \xi)\mathcal{T}_5(\alpha + \beta)\mathcal{T}_6(\gamma)d\beta d\gamma d\eta d\xi dt
$$

Next we use the Fourier inversion formula $\int_{\mathbb{R}} \int_{\mathbb{R}} e^{iat\phi} \phi(x) dxdt = 2\pi |a|^{-1} \phi(0)$, with $a = 2\beta$ and $x(\alpha) = \xi + \gamma\lambda - \xi\lambda - \alpha$. This gives,

$$
E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^2} \int_{\mathbb{R}^6} f_1(\beta + \xi)f_2(\lambda\beta + \gamma)f_3(\xi + \gamma\lambda - \xi\lambda)
$$

$$
\mathcal{T}_4(\lambda\beta + \xi)\mathcal{T}_5(\beta + \xi + \lambda\gamma - \xi\lambda)\mathcal{T}_6(\gamma)d\beta d\gamma d\eta d\xi
$$

which is (4.18). The representation (4.19) follows from the definition of $\mathcal{T}_6$ in (4.6) with the $L^2$ inner product integration in $\gamma$.

\section*{Theorem 4.6.} There holds the representations

\begin{equation}
E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \frac{1}{\lambda^2 - \lambda + 1} E_{A(\lambda)}(f_1, f_2, f_3, f_4, f_5, f_6)d\lambda,
\end{equation}

\begin{equation}
\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \frac{1}{\lambda^2 - \lambda + 1} T_{A(\lambda)}(f_1, f_2, f_3, f_4, f_5)(x)d\lambda.
\end{equation}

where, for all $\lambda$, $A(\lambda)$ is an isometry and $A(\lambda)(1, 1, 1) = (1, 1, 1)$. (The matrix $A(\lambda)$ is given explicitly in (4.22).)

Proof. In formula (4.18), let $y_1, y_2, y_3$ be the arguments of $f_1, f_2, f_3$ respectively, and let $x_1, x_2, x_3$ be the arguments of $\mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6$ respectively. We have

$$
y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \beta + \xi \\ \lambda\beta + \gamma \\ \lambda\gamma - \xi\lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \lambda & 1 - \lambda \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix} := B(\lambda) \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix}
$$

and

$$
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda\beta + \xi \\ \beta + \lambda\gamma - \xi\lambda - \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & 1 & 1 - \lambda \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix} := C(\lambda) \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix}.
$$

In equation (4.18), perform the linear change of variables $x = C(\lambda)(\beta, \gamma, \xi)$. We find that $\det C(\lambda) = \lambda^2 - \lambda + 1 = (\lambda - 1)^2 + \frac{3}{4} > 0$; in particular $C(\lambda)^{-1}$ is defined for all $\lambda$. Let $A(\lambda) = B(\lambda)C(\lambda)^{-1}$. Changing variables then establishes (4.20). The expression for $\mathcal{T}_6$ follows using the definition of $T_A$ in Definition 3.1.

A calculation reveals that,

\begin{equation}
A(\lambda) = B(\lambda)C(\lambda)^{-1} = \frac{1}{\lambda^2 - \lambda + 1} \begin{pmatrix} \lambda & 1 - \lambda & \lambda^2 - \lambda \\ \lambda^2 - \lambda & \lambda & 1 - \lambda \\ 1 - \lambda & \lambda^2 - \lambda & \lambda \end{pmatrix}
\end{equation}

\begin{equation}
= \frac{1}{\lambda^2 - \lambda + 1} \begin{pmatrix} \lambda & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \lambda(\lambda - 1) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\end{equation}
It remains to verify the two properties of $A(\lambda)$. These can, of course, be determined from the formula (4.22); however it is more didactic to see how they arise naturally from the combinatorial structure of the arguments to the functions in (4.18).

(i) By inspecting (4.18), we find that the squares of the arguments in $f_1, f_2, f_3$ sum to the squares of the arguments in $f_4, f_5, f_6$,

\[(\beta + \xi)^2 + (\lambda \beta + \gamma)^2 + (\lambda \gamma + \xi - \lambda \xi)^2 = (\lambda \beta)^2 + (\beta + \lambda \gamma + \xi - \lambda \xi)^2 + (\gamma)^2.\]

This gives, for all $x \in \mathbb{R}^3$, that $|B(\lambda)x|^2 = \sum_{k=1}^3 |(B(\lambda), e_k)|^2 = \sum_{k=1}^3 |(C(\lambda), e_k)|^2 = |C(\lambda)x|^2$. Setting $x = C(\lambda)^{-1}y$ gives $|A(\lambda)y|^2 = |y|^2$ for all $y \in \mathbb{R}^3$, and hence $A(\lambda)$ is an isometry.

(ii) Again in (4.18), we see that the arguments in $f_1, f_2, f_3$ sum to the arguments in $f_4, f_5, f_6$,

\[(\beta + \xi) + (\lambda \beta + \gamma) + (\lambda \gamma + \xi - \lambda \xi) = (\lambda \beta) + (\beta + \lambda \gamma + \xi - \lambda \xi) + (\gamma).\]

Setting $e = (1, 1, 1)$, this means that for all $x$, $\langle B(\lambda)x, e \rangle = C(\lambda)x, e \rangle$. Set $y = C(\lambda)x$ to give $\langle A(\lambda)y, e \rangle = \langle y, e \rangle$. Because $A$ is an isometry, $A^* = A^{-1}$, and so $\langle y, A^{-1}e \rangle = \langle y, e \rangle$ for all $y$, and hence $Ac = e$.

We note that the expressions (4.24) and (4.25) arise naturally from the $\delta$ arguments in (4.14). The properties of $A(\lambda)$ in (i) and (ii) should thus be considered generic for continuous resonant type equations. □

**Theorem 4.7.** We have the representations,

\[
\mathcal{E}(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\sqrt{3} \pi^2} \int_0^{2\pi} E_{R(\theta)}(f_1, f_2, f_3, f_4, f_5, f_6) d\theta,
\]

\[
T(f_1, f_2, f_3, f_4, f_5)(x) = \frac{\sqrt{3}}{\pi^2} \int_0^{2\pi} T_{R(\theta)}(f_1, f_2, f_3, f_4, f_5)(x) d\theta,
\]

where $R(\theta)$ is the rotation of $\theta$ radians about the axis $(1,1,1)$.

**Proof.** Because the matrix $A(\lambda)$ is an isometry, $\det A(\lambda) = +1$, and $A(\lambda)(1,1,1) = (1,1,1)$, the matrix must, in fact, be a rotation about the axis $(1,1,1)$. For any rotation $A$ of $\mathbb{R}^3$, the angle of rotation $\theta$ satisfies, $2 \cos(\theta) + 1 = \text{Trace}(A)$. In the present case, this means,

\[
\cos(\theta) = \phi(\lambda) := \frac{1}{2} \left( \text{Trace}(A(\lambda)) - 1 \right) = \frac{1}{2} \left( \frac{3\lambda}{\lambda^2 - \lambda + 1} - 1 \right).
\]

The formula (4.26) follows from performing the change of variables $\lambda \mapsto \theta$, which we do rather carefully.

From analyzing (4.28), we determine that $\phi$ has the following properties. It satisfies $\phi(-1) = -1$, $\phi(1) = 1$; $\phi$ is increasing on $[-1,1]$; $\phi$ is decreasing on $(-\infty, -1] \cup [1, \infty)$; and $\lim_{\lambda \to -\infty} \phi(\lambda) = \lim_{\lambda \to +\infty} \phi(\lambda) = 0$. By setting $\lambda = 0$ in (4.23), we determine that $\sin(\theta) = -1\sqrt{3} < 0$, and hence $\theta = -4\pi/3 \in [\pi, 2\pi]$. On the other hand, as $\lambda \to +\infty$, $\sin(\theta) \to +1/\sqrt{3} > 0$. From these considerations and continuity, we infer that that under $\lambda \mapsto \theta$, $[-1,1]$ is bijectively mapped to $[\pi, 2\pi]$, while $(-\infty, -1) \cup (1, \infty)$ is bijectively mapped to $(0, \pi)$. In all, $\mathbb{R}$ is bijectively mapped to $(0, 2\pi]$.

To perform the change of variables, we need to find the determinant which is given by,

\[
\frac{d\theta}{d\lambda} = \left| \frac{d}{d\lambda} \arccos \left( \frac{1}{2} \left[ \frac{3\lambda}{\lambda^2 - \lambda + 1} - 1 \right] \right) \right|.
\]

To simplify the computation, we first find that if $a = \sqrt{3}(1 - \lambda)/(1 + \lambda)$, then

\[
\arccos \left( \frac{1}{2} \left[ \frac{3\lambda}{\lambda^2 - \lambda + 1} - 1 \right] \right) = \arccos \left( \frac{1 - a^2}{1 + a^2} \right) = \arctan \left( \frac{2a}{1 - a^2} \right)
\]

\[= 2 \arctan(a) = 2 \arctan \left( \frac{\sqrt{3}(1 - \lambda)}{1 + \lambda} \right);
\]

we then calculate $|d\theta/d\lambda| = \sqrt{3}(\lambda^2 - \lambda + 1)^{-1}$. Formula (4.26) then follows. □
We have derived, in total, six different representations for each of \( \mathcal{E}_6, \mathcal{H}_6 \) and \( \mathcal{T}_6 \). Before moving on, we give a sketch of the relevance of each of these representations. The first expression (4.2), in terms of \( e^{itH} \), shows that the Hamiltonian system \( \mathcal{H}_6 \) is the resonant system for \( k = 2 \) discussed in Section 2. It also illustrates clearly the permutation symmetries (4.3) and (4.4) which are far from obvious from the last three representations in Theorems 4.5, 4.6 and 4.7. The representations (4.12) and (4.14) will not be used extensively in this article, but they very importantly show that \( \mathcal{H}_d \) is the continuous resonant system for \( d = 1 \) and \( p = 2 \) discussed in [8]. In the remainder of this section we will mostly use the last three representations. The expression (4.18) is useful for proving refined multilinear estimates: if we have information about the supports of the functions \( f_k \) then we can restrict the range of integration and prove smallness, in tandem with (4.20). The last representation (4.26) makes determining certain properties of \( \mathcal{E}_6 \), like \( L^2 \) boundedness and symmetry structures, almost trivial because we know so much about \( E_{R(\theta)} \) from Section 3. The following two subsections are largely about importing such information about \( E_A \) from Section 3 to \( \mathcal{E}_6 \).

4.2. Symmetries of the Hamiltonian and conserved quantities of the flow.

**Theorem 4.8.** The functional \( \mathcal{E}_A(f_1, f_2, f_3, f_4, f_5, f_6) \) is invariant under the following actions (for all \( \lambda \)).

(i) Fourier transform, \( f_k \mapsto \hat{f}_k \).

(ii) Modulation, \( f_k \mapsto e^{i\lambda} f_k \).

(iii) \( L^2 \) scaling, \( f_k(x) \mapsto \lambda^{1/2} f_k(\lambda x) \).

(iv) Linear modulation, \( f_k \mapsto e^{i\lambda} f_k \).

(v) Translation, \( f_k \mapsto f_k(\cdot + \lambda) \).

(vi) Quadratic modulation, \( f_k \mapsto e^{i\lambda x^2} f_k \).

(vii) Schrödinger group, \( f_k \mapsto e^{i\lambda \Delta} f_k \).

(viii) Schrödinger with harmonic trapping group, \( f_k \mapsto e^{i\lambda H} f_k \).

**Proof.** Each of the symmetries commutes with the integration over \( \theta \) in (4.26). The result then follows immediately from Theorems 3.2 and 3.3. \( \square \)

**Corollary 4.9.** We have the following commuter equalities,

\[
(4.29) \quad e^{i\lambda Q} \mathcal{T}(f_1, f_2, f_3, f_4, f_5) = \mathcal{T}(e^{i\lambda Q} f_1, e^{i\lambda Q} f_2, e^{i\lambda Q} f_3, e^{i\lambda Q} f_4, e^{i\lambda Q} f_5)
\]

\[
Q \mathcal{T}(f_1, f_2, f_3, f_4, f_5) = \mathcal{T}(Q f_1, f_2, f_3, f_4, f_5) + \mathcal{T}(f_1, Q f_2, f_3, f_4, f_5) + \mathcal{T}(f_1, f_2, Q f_3, f_4, f_5) - \mathcal{T}(f_1, f_2, f_3, f_4, Q f_5)
\]

(4.30)

where \( Q \) are the operators: \( Q = 1, Q = x, Q = \text{id}/\text{dx}, Q = x^2, Q = \Delta, Q = H \).

**Proof.** These follow immediately from the representation (4.27) and Theorem 3.4. \( \square \)

By Theorem 3.5, the Hamiltonian flow associated to \( \mathcal{H}_6 \) has conserved quantities for symmetries (ii) through (viii). These symmetries and conserved quantities are summarized in the following table.

<table>
<thead>
<tr>
<th>Symmetry of ( \mathcal{H}_6 )</th>
<th>Conserved quantity</th>
<th>Operator commuting with ( \mathcal{T}_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f \mapsto e^{i\lambda f} )</td>
<td>( \int_R</td>
<td>f(x)</td>
</tr>
<tr>
<td>( f \mapsto f_{\lambda} )</td>
<td>( \int_R [xf'(x) + f(x)] \mathcal{F}(x) \text{dx} )</td>
<td>( x )</td>
</tr>
<tr>
<td>( f \mapsto e^{i\lambda x} f )</td>
<td>( \int_R x</td>
<td>f(x)</td>
</tr>
<tr>
<td>( f \mapsto f(\cdot + \lambda) )</td>
<td>( \text{Re} \int_R f(x) \mathcal{F}(x) \text{dx} )</td>
<td>( \text{id}/\text{dx} )</td>
</tr>
<tr>
<td>( f \mapsto e^{i\lambda</td>
<td>x</td>
<td>^2} f )</td>
</tr>
<tr>
<td>( f \mapsto e^{i\lambda \Delta} f )</td>
<td>( \int_R</td>
<td>f'(x)</td>
</tr>
<tr>
<td>( f \mapsto e^{i\lambda H} f )</td>
<td>( \int_R</td>
<td>xf(x)</td>
</tr>
</tbody>
</table>
4.3. Boundedness of the functional and wellposedness of Hamilton’s equation.

**Theorem 4.10.** There holds the following sharp bound,

\[
|E_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{\pi \sqrt{3}} \prod_{k=1}^{6} \|f_k\|_{L^2};
\]

which means in particular \(0 \leq \mathcal{H}_6(f) \leq 1/(\pi \sqrt{3})\|f\|_{L^2}^6\). Equality holds in (4.31) if and only if each \(f_k\) is the same Gaussian \(ce^{-\alpha x^2+\beta x}\) for some \(c, \alpha, \beta \in \mathbb{C}\) and \(\Re \alpha > 0\).

**Proof.** Using representation (4.26) and Theorem 3.6 we have,

\[
|E_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} |E_R(\theta)(f_1, f_2, f_3, f_4, f_5, f_6)|d\theta \leq \frac{1}{\pi \sqrt{3}} \prod_{k=1}^{6} \|f_k\|_{L^2},
\]

which is the inequality (4.31).

For equality to hold, we must have equality in,

\[
|E_R(\theta)(f_1, f_2, f_3, f_4, f_5, f_6)| = \prod_{k=1}^{6} \|f_k\|_{L^2}
\]

for almost every \(\theta \in [0, 2\pi]\). We know from Theorem 3.12 that if the functions \(f_k\) are the same Gaussian \(e^{-\alpha x^2+\beta x}\) for \(\alpha, \beta \in \mathbb{C}\) and \(\Re \alpha > 0\) then equality does hold. (This uses the fact that \(R(\theta)\) is an isometry for all \(\theta\) and that \(R(\theta)(1, 1, 1) = (1, 1, 1)\).)

On the other hand, because \(R(\theta)\) is not a signed permutation matrix almost everywhere, Theorem 3.12 says that any functions \(f_k\) satisfying (4.32) must be Gaussians. Write \(f_k(x) = e^{-\alpha_k x^2+\beta_k x}\) for \(\alpha_k, \beta_k \in \mathbb{C}\) with \(\Re \alpha_k > 0\). The Cauchy–Schwarz equality condition (3.19) in this case reads,

\[
f_1((R(\theta)x)_1)f_2((R(\theta)x)_2)f_3((R(\theta)x)_3) = f_1(x_1)f_2(x_2)f_3(x_3),
\]

Substituting the expressions for \(f_k\) gives, for all \(\theta \in [0, 2\pi]\) and \((x_1, x_2, x_3) \in \mathbb{R}^3\),

\[
\sum_{k=1}^{3} -\alpha_k(R(\theta)x)_k^2 + \beta_k(R(\theta)x)_k = \sum_{k=1}^{3} -\alpha_1 x_k^2 + \beta_1 x_k.
\]

Setting variously \(x = (1, 0, 0)\), \(x = (0, 1, 0)\) and \(x = (0, 0, 1)\), along with \(\theta = 0\), \(\theta = 2\pi/3\) and \(\theta = 4\pi/3\) gives that \(\alpha_k = \alpha_1\) and \(\beta_k = \beta_1\) for all \(k\).

Using the expression \(\mathcal{H}_6(f) = (2/\pi)\|e^{iHt}f\|_{L^6}^6\) from Theorem 4.3, one obtains the sharp inequality,

\[
\|e^{iHt}f\|_{L^6}^6 \leq \frac{1}{2\sqrt{3}} \|f\|_{L^2}^6,
\]

which is the Strichartz inequality with best constant in dimension one, as previously proved in [13]

**Theorem 4.11.** We have the operator bound \(\|T_6(f_1, f_2, f_3, f_4, f_5)\|_{X} \leq C_X \prod_{k=1}^{5} \|f_k\|_{X}\), for the following spaces.

(i) \(X = L^2\) with \(C_X = 2\sqrt{3}/\pi\).

(ii) \(X = L^{2,\sigma}\), for any \(\sigma \geq 0\).

(iii) \(X = H^\sigma\), for any \(\sigma \geq 0\).

(iv) \(X = L^{\infty^*, s}\), for any \(s > 1/2\).

(v) \(X = L^{p, s}\), for any \(p \geq 2\) and \(s > 1/2 - 1/p\).

**Proof.** The bounds (i), (ii), and (iii) follow immediately from Theorem 3.7 using the representation of \(T_6\) in (4.27).

The bound (iv) is proved in Theorem 4.18 below.

The bound (v) comes from interpolating between the bounds in (iv) and (ii).
Theorem 4.12. Consider the Cauchy problem,
\begin{align}
    iu_t &= T_6(u, u, u, u, u), \\
    u(t = 0) &= u_0,
\end{align}
which is Hamilton’s equation corresponding to $\mathcal{H}_6$ and the resonant equation (1.4) in the quintic $k = 2$ case.

(i) The Cauchy problem (4.35) is locally well-posed in $X$ for any of the spaces $X$ in Theorem 4.11. (ii) The Cauchy problem (4.35) is globally well-posed in $L^2$.

\textbf{Proof.} (i) This follows from Theorem 3.8, using the bounds on $T_6$ established in the previous theorem.

(ii) We know from Theorem 3.8 that the local time of existence of a solution to (4.35) in $L^2$ depends only on $\|u_0\|_{L^2}$. Because $\|u\|_{L^2}$ is conserved by the flow (4.35), by the usual argument the $L^2$ solution is global.

\hfill $\blacksquare$

4.4. Analysis of the stationary waves. Stationary wave solutions are solutions of the form $e^{i\omega t}\psi(x)$ for some $\omega \in \mathbb{R}$ and a function $\psi$. By substitution into (4.5), we find that $\psi$ must satisfy
\begin{equation}
    -\omega \psi(x) = T_6(\psi, \psi, \psi, \psi, \psi)(x) = \frac{\sqrt{3}}{\pi^2} \int_0^{2\pi} T_R(\theta)(\psi, \psi, \psi, \psi)(x)d\theta.
\end{equation}

In Theorem 3.10, we found that if $\phi_n$ is a Hermite function then $T_A(\phi_n, \phi_n, \phi_n, \phi_n, \phi_n)(x) = C\phi_n(x)$, which means from (4.36) that $T_6(\phi_n, \ldots, \phi_n)(x) = C\phi_n(x)$ for some constant $C$. The function $\phi_n$ is thus a stationary wave. By letting the symmetries of $T_6$ act on $\phi_n$, we find that each of the functions
\begin{equation}
    ae^{ibx + icx^2}\phi_n(dx + e),
\end{equation}
for $a \in \mathbb{C}$ and $b, c, d, e \in \mathbb{R}$ is a stationary wave solution of (4.5).

4.4.1. Regularity of stationary waves: introduction. All of the stationary waves (4.37) are analytic and decay in space like $e^{-\alpha x^2}$ for some $\alpha \in \mathbb{R}$. The remainder of this section is devoted to a proof any function $\psi \in L^2$ satisfying (4.36) is automatically analytic and exponentially decaying in space like $e^{-\alpha x^2}$. Our proof follows closely the proof of the analogous result for the two-dimensional continuous resonant equation in [17], which in turn is based on work in [22]; there are also similar results in [11, 18].

Our proof here has two main ingredients. Roughly speaking, once a multilinear functional can supply these ingredients, the associated Hamiltonian system will satisfy a result like Theorem 4.15 below. The first ingredient is an ability to transfer exponential weight from one input of the functional to the other inputs, as in Lemma 3.13 for the functionals $E_A$. The second ingredient is a refined multilinear estimate, which we discuss in the next subsection.

For the weight transfer property, recall that in Lemma 3.13, we defined $G_{\mu, r}(x) = \exp(\mu x^2/(1 + r^2))$ and established that if $A$ is an isometry and functions $f_k$ are non-negative, then,
\begin{equation}
    E_A(f_1, \ldots, f_{2n-1}, f_{2n} G_{\mu, r}) \leq E_A(f_1 G_{\mu, r}, \ldots, f_{2n-1} G_{\mu, r}, f_{2n}).
\end{equation}
This property is immediately inherited by $\mathcal{E}_6$: if functions $f_k$ are non-negative, then,
\begin{align}
    \mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6 G_{\mu, r}) &= \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_R(\theta)(f_1, f_2, f_3, f_4, f_5, f_6 G_{\mu, r})d\theta \\
    &\leq \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_R(\theta)(f_1 G_{\mu, r}, f_2 G_{\mu, r}, f_3 G_{\mu, r}, f_4 G_{\mu, r}, f_5 G_{\mu, r}, f_6) d\theta \\
    &= \mathcal{E}_6(f_1 G_{\mu, r}, f_2 G_{\mu, r}, f_3 G_{\mu, r}, f_4 G_{\mu, r}, f_5 G_{\mu, r}, f_6).
\end{align}
The inequality (4.38) is the first ingredient in the proof.
4.4.2. Refined multilinear Strichartz estimates. The second ingredient we need is a so-called refined multilinear Strichartz estimate. Such estimates are treated in a number of works [3, 5, 23]. Lemma 111 in [5] is prototypical of the type of estimate we require here: it states that if functions $f_1, f_2 \in L^2(\mathbb{R}^2 \to \mathbb{C})$ satisfy $\text{supp} \hat{f}_1 \subset B(0, N)$ and $\text{supp} \hat{f}_2 \subset B(0, M)^C$, with $N \ll M$, then,

$$\| (e^{it\Delta} f_1)(e^{it\Delta} f_2) \|_{L^4(\mathbb{R}^2 \times \mathbb{R})} \lesssim \left( \frac{N}{M} \right)^{1/2} \| f_1 \|_{L^2(\mathbb{R}^2)} \| f_2 \|_{L^2(\mathbb{R}^2)}.$$ 

The right hand side is decaying for large $M$ and small $N$. In our case, under similar support assumptions on functions $\hat{f}_i$ and $\hat{f}_j$, we would like to have analogous control on,

$$E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi} \int_\mathbb{R} \int_\mathbb{R} (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5)(e^{it\Delta} f_6) dx dt;$$

namely, we would like an $L^2$ bound that is decaying as the supports of $\hat{f}_i$ and $\hat{f}_j$ become increasingly disjoint. Using the representations (4.18) and (4.20) we are in fact able to determine the required refined multilinear estimate in an elementary way.

Because we know that $E_6$ is invariant under the Fourier transform, it is equivalent to state the support assumptions in terms of $f_i$ and $f_j$ and not their Fourier transforms.

**Proposition 4.13.** Suppose that the support of $f_2$ is in $B(0, R)^C$ and the supports of $f_3, f_5$ and $f_6$ are in $B(0, r)$, with $R > 4r$. Then

$$|E_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{R} \prod_{k=1}^{6} \| f_k \|_{L^2}.$$

**Proof.** We use the representation of $E_6$ given in (4.18),

$$E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^2} \int_{\mathbb{R}^6} f_1(\beta + \xi) f_2(\lambda \beta + \gamma) f_3(\lambda \gamma + \xi - \lambda \xi) f_4(\lambda \beta + \xi) f_5(\lambda \gamma + \xi - \lambda \xi) f_6(\gamma) d\beta d\eta d\xi d\gamma,$$

to identify a large set in $\lambda$ on which the integrand is 0. We will then use the representation (4.20) to obtain $L^2$ bounds, recalling that the integrand as a function of $\lambda$ is the same in both representations.

Under the assumptions of the proposition, the integrand is non-zero only when

$$|\beta| \leq |\beta + \lambda \gamma + \xi - \lambda \xi| + |\lambda \gamma + \xi - \lambda \xi| \leq 2r,$$

and only when,

$$|\lambda \beta| \geq |\lambda \beta + \gamma| - |\gamma| \geq R - 2r \geq \frac{R}{2}.$$ 

It follows that the integrand is non-zero only when $|\lambda| > R/4$.

Then, using the representation (4.20),

$$|E_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{2\pi^2} \int_{|\lambda| > R/4} \frac{1}{\lambda^2 - \lambda + 1} |E_{A(\lambda)}(f_1, f_2, f_3, f_4, f_5, f_6)| d\lambda \leq \frac{1}{\pi^2} \left( \int_{|\lambda| > R/4} \frac{1}{\lambda^2 - \lambda + 1} d\lambda \right) \prod_{k=1}^{6} \| f_k \|_{L^2} \leq \frac{1}{R} \prod_{k=1}^{6} \| f_k \|_{L^2},$$

which is what we wanted to prove. 

**Proposition 4.14.** Suppose that for some $i$ and some $j$, the support of $f_i$ is in $B(0, R)^C$ and the support of $f_j$ is in $B(0, r)$, with $R > 4r$. Then

$$|E_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{R^{1/6}} \prod_{k=1}^{6} \| f_k \|_{L^2}. (4.39)$$
Proof. We assume, by rescaling, that \(|f_k|_{L^2} = 1\) for all \(k\). We have the crude bound \(H_0(f) = \|e^{it\Delta} f_k\|_{L^6}^6 \leq 1\), from (4.34). Then,

\[
|\mathcal{E}_0(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{2}{\pi} \int_{\mathbb{R}^2} |e^{it\Delta} f_1| \cdot |e^{it\Delta} f_2| \cdot |e^{it\Delta} f_3| \cdot |e^{it\Delta} f_4| \cdot |e^{it\Delta} f_5| \cdot |e^{it\Delta} f_6| \, dx dt,
\]

Taking the limit as \(\epsilon \to 0\), we have

\[
\frac{2}{\pi} \| |(e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5)(e^{it\Delta} f_6)| \|_{L^3}
\]

\[
= \frac{2}{\pi} \| |(e^{it\Delta} f_1)^3(e^{it\Delta} f_3)^3| |_{L^{1/3}} \|
\]

\[
\leq \frac{2}{\pi} \| |(e^{it\Delta} f_1)(e^{it\Delta} f_3)^2| |_{L^{5/3}} \|
\]

\[
\leq \left( \frac{2}{\pi} \right)^{2/3} \mathcal{E}_0(f_1, f_2, f_3, f_4, f_5, f_6)^{1/6} \leq \frac{1}{R^{1/6}},
\]

which is (4.39). \(\square\)

4.4.3. Regularity of stationary waves. Using the weight transfer property (4.38) and the refined multilinear Strichartz estimate (4.39), we prove that stationary waves are necessarily analytic. We begin with an integrability result.

Theorem 4.15. Suppose \(\phi \in L^2\) satisfies

\[
|\omega|\phi(x)| \leq T(|\phi|, |\phi|, |\phi|, |\phi|, |\phi|)(x).
\]

Then there exists \(\alpha > 0\) such that \(x \mapsto \phi(x)e^{\alpha x^2} \in L^2\).

Proof of Theorem 4.15. For the proof, we will find \(\mu\) so that we have the bound \(\|\phi G_{\mu, \epsilon}\|_{L^2} \lesssim 1\) independently of \(\epsilon\). Taking the limit \(\epsilon \to 0\) will yield the result.

We can clearly assume that \(\phi(x) \geq 0\), and will do so throughout. For any \(M > 0\) define,

\[
\phi_\epsilon(x) = \phi(x) \chi_{|x| \leq M}(x), \quad \phi_\epsilon(x) = \phi(x) \chi_{|x| \leq M^2}(x), \quad \phi_\epsilon(x) = \phi(x) \chi_{M^2 < |x|}(x),
\]

We have the decomposition \(\phi = \phi_\epsilon + \phi_\epsilon + \phi_\epsilon + \phi_\epsilon\), and the supports are all disjoint, which gives,

\[
\|\phi G_{\mu, \epsilon}\|_{L^2} = \|\phi_\epsilon G_{\mu, \epsilon}\|_{L^2} + \|\phi_\epsilon G_{\mu, \epsilon}\|_{L^2} + \|\phi_\epsilon G_{\mu, \epsilon}\|_{L^2}.
\]

The first two terms are trivial to deal with. If \(|x| \leq M^2\), we have,

\[
G_{\mu, \epsilon}(x) = \exp(\mu |x|^2) \leq \exp(\mu M^4),
\]

so setting \(\mu \leq M^{-4}\) gives \(\|\phi_\epsilon G_{\mu, \epsilon}\|_{L^2} \leq \|\phi_\epsilon G_{\mu, \epsilon}\|_{L^2} \leq \epsilon\|\phi\|_{L^2}\), uniformly in \(\epsilon\). The same bound holds for \(\phi_\epsilon\). It remains then to bound \(\|\phi G_{\mu, \epsilon}\|_{L^2}\) uniformly in \(\epsilon\).

Starting with equation (4.40), we multiply both sides by \(\phi_\epsilon(x) G_{\mu, \epsilon}(x)^2\),

\[
\omega \phi_\epsilon(x) G_{\mu, \epsilon}(x)^2 \leq T(\phi_\epsilon, \ldots, \phi_\epsilon)(x)\phi_\epsilon(x) G_{\mu, \epsilon}(x)^2.
\]

Now integrating over \(\mathbb{R}\), using the relationship between \(\mathcal{E}_0\) and \(\mathcal{T}_0\) in (4.6), and passing the exponential weight using (4.38), we determine the bound,

\[
\omega \|\phi G_{\mu, \epsilon}\|_{L^2}^2 \leq 6\mathcal{E}_0(\phi, \phi, \phi, \phi, \phi, \phi G_{\mu, \epsilon}^2) \leq \mathcal{E}_0(\phi G_{\mu, \epsilon}^3, \ldots, \phi G_{\mu, \epsilon}^3, \phi G_{\mu, \epsilon}^3).
\]

For convenience, let \(\psi = \phi G_{\mu, \epsilon}\). The bound then reads,

\[
\omega \|\psi\|_{L^2}^2 \leq \mathcal{E}_0(\psi, \psi, \psi, \psi, \psi, \psi).
\]

Now write each \(\psi = \psi_\epsilon + \psi_\epsilon + \psi_\epsilon\) and expand the multilinear functional. We will get many terms, which we bound in one of two ways.

- If there are three or more \(\psi_\epsilon\) terms, bound by \(\|\psi_\epsilon\|_{L^2}^k\) (where \(k \geq 3\) is the number of \(\psi_\epsilon\) terms appearing) using the standard \(L^2\) bound (4.31). In this case the other terms are \(\psi_\epsilon\) or \(\psi_\epsilon\), which we know are uniformly bounded.
- If there are one or two \(\psi_\epsilon\) terms, then there is either a \(\psi_\epsilon\) term or a \(\psi_\epsilon\) term. We may assume \(M > 4\). Then in the former case we can use the refined multilinear estimate (4.39) (with \(r = M\) and \(R = M^2\) and bound by \((1/M^{1/3})\|\psi_\epsilon\|_{L^2}^k\) (where \(k = 1\) or \(k = 2\)). If there are no \(\psi_\epsilon\) terms, we bound by \(\|\psi_\epsilon\|_{L^2}^k\|\psi_\epsilon\|_{L^2}^k \leq \|\psi_\epsilon\|_{L^2}^k\|\psi_\epsilon\|_{L^2}^k\).
Using these, we find,

\[(4.42)\]

\[
\omega \| \psi_\mu \|_{L^2}^2 \leq 6E_0(\psi, \psi, \psi, \psi, \psi_\mu) \leq C \left( \sum_{k=3}^{m} \| \psi_\mu \|^{k} + \left( \frac{1}{M^{1/3}} + \| \phi_\mu \|_{L^2} \right) \left( \| \psi_\mu \|_{L^2} + \| \psi_\mu \|_{L^2} \right) \right),
\]

for some constant \(C\) independent of \(\psi\) and \(\epsilon\). Set \(\delta(M) = M^{-1/3} + \| \phi_\mu \|_{L^2}\) and

\[
x(\epsilon, M) = \| \psi_\mu \|_{L^2} = \| \phi_\chi |x| < MzG_{\mu, x} \|_{L^2}.
\]

Note that \(\delta(M) \to 0\) as \(M \to \infty\). Choose \(M\) sufficiently large so that \(C\delta(M) \leq \omega/2\). This gives,

\[
\frac{\omega}{2C} e(\epsilon, M)^2 \leq \sum_{k=3}^{m} x(\epsilon, M)^k + \delta(M)x(\epsilon, M).
\]

Dividing through by \(x(\epsilon, M) > 0\) and rearranging terms gives

\[(4.43)\]

\[0 \leq p_{\delta(M)}(x(\epsilon, M)), \quad \text{where} \quad p_\delta(x) := \sum_{k=2}^{m-1} x^k - \frac{\omega}{2C}x + \delta.
\]

Observe that \(p_0(0) = 0, p_0'(0) = -\omega/2C < 0\) and \(p_0(x) \to \infty\) as \(x \to \infty\). This shows that \(p_0\) has another 0 in \((0, \infty)\); call the smallest such zero \(x_0\). The zeroes of a polynomial are continuous functions of the coefficients. Hence if we choose \(M\) sufficiently large we can assume that \(p_{\delta(M)}\) has one zero in \((-\infty, x_0/3)\) (coming from \(p_0(0) = 0\)) and one zero \((2x_0/3, \infty)\) (coming from \(p_0(x_0) = 0\)) and that \(p_{\delta(M)}(x) < 0\) in \((x_0/3, 2x_0/3)\). This shows that for all \(M\) sufficiently large,

\[
Z_{\delta(M)} = p_{\delta(M)}^{-1}(0, \infty)) \subseteq (-\infty, x_0/3) \cup (2x_0/3, \infty).
\]

Now we know from the inequality (4.43) that \(x(\epsilon, M) \in Z_{\delta(M)}\) for all \(\epsilon\). If we set \(\epsilon = 1\) we get,

\[
x(1, M) = \| \psi_\mu \|_{L^2} = \| \phi_\mu e^{\mu x^2/(1+x^2)} \|_{L^2} \leq \| \phi_\mu \|_{L^2} e^\mu.
\]

Recall that we set \(\mu = M^{-1}\), so that \(\mu \leq 1\) and so \(x(1, M) \lesssim \| \phi_\mu \|_{L^2}\). As \(M \to \infty\), \(\| \phi_\mu \|_{L^2} = \| \phi_\chi_{M^2 < |x|} \|_{L^2} \to 0\), and hence if we take \(M\) sufficiently large we will have \(x(1, M) \leq x_0/3\). But now because \(x(\epsilon, M)\) depends continuously on \(\epsilon\), and \(x(\epsilon, M) \in Z_{\delta(M)}\) for all \(\epsilon\), we have,

\[
x(\epsilon, M) = \| \psi_\mu \|_{L^2} \leq x_0/3,
\]

for all \(\epsilon\). Taking \(\epsilon \to 0\) yields \(x(0, M) = \| \phi_\mu e^{\mu x^2} \| \leq x_0/3 < \infty\), which is what we wanted to prove. \(\square\)

**Corollary 4.16.** Suppose that \(\phi \in L^2\) is a stationary wave solution of the Hamiltonian flow associated to \(H_0\); that is, \(\phi\) satisfies

\[(4.44)\]

\[
\omega \phi(x) = T_\phi(\phi, \phi, \phi, \phi, \phi)(x),
\]

for some \(\omega\). Then there exists \(\alpha > 0\) and \(\beta > 0\) such that \(\phi e^{\alpha x^2} \in L^\infty\) and \(\phi e^{\beta x^2} \in L^\infty\). As a result, \(\phi\) can be extended to an entire function on the complex plane.

**Proof.** The condition (4.44) implies the condition (4.40) in the previous theorem, and hence there exists \(\alpha > 0\) such that \(\phi e^{\alpha x^2} \in L^2\). Because \(T_\phi\) commutes with the Fourier transform, condition (4.44) also holds with \(\phi\) replaced by \(\hat{\phi}\). Then, again by the previous theorem, there exists \(\beta > 0\) such that \(\phi e^{\beta x^2} \in L^2\).

To turn these \(L^2\) bounds into \(L^\infty\) bounds, we first assume \(\phi\) is Schwartz and compute,

\[
\phi(x)^2 e^{2\alpha x^2} = e^{2\alpha x^2} \int_x^\infty \frac{d}{dt} \phi(t)^2 dt = e^{2\alpha x^2} \int_x^\infty 2\phi(t) \phi'(t) dt \leq 2 \int_x^\infty e^{2\alpha t^2} \phi(t) \phi'(t) dt 
\leq 2 \| e^{2\alpha t^2} \phi(t) \|_{L^2} \| \phi'(t) \|_{L^2} = 2 \| e^{2\alpha t^2} \phi(t) \|_{L^2} \| \xi \phi(t) \|_{L^2} \leq \beta^{-1/2} \| e^{2\alpha t^2} \phi(t) \|_{L^2} \| e^{2\beta t^2} \phi(t) \|_{L^2},
\]

which gives \(\phi(x)e^{\alpha x^2} \in L^\infty\). Because Schwartz functions are dense in \(L^2\), this holds for arbitrary \(\phi \in L^2\). The \(L^\infty\) bound for \(\phi\) follows similarly.
Finally, using the $L^\infty$ bound $|\hat{\phi}(\xi)| \leq e^{-\beta \xi^2}$, and the inverse Fourier transform formula,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iz\xi} \hat{\phi}(\xi) d\xi,$$

we can extend $\phi$ to an entire function on the complex plane.

\[\square\]

4.5. **Smoothing and further boundedness properties.** In this section we prove two further boundedness results for the operator $T_\sigma$. Unlike our previous boundedness results, which were simply inherited from the analogous results for $T_A$, the present results rely on additional structure in $T_\sigma$.

We first strengthen items (ii) and (iii) in Theorem 4.11. Item (ii) says that $T_\sigma$ is bounded from $L^2(\sigma)$ to $L^2(\sigma)$ for some $\delta > 0$. By Fourier invariance, $T_\sigma$ maps $(H^s)\delta$ to $H^{\sigma+\delta}$.

The second result here establishes boundedness of $T_\sigma$ from $(L^\infty,s)\delta$ to $L^\infty,s$ for any $s > 1/2$. The analogous result for $s < 1/2$ is false, by scaling. We conclude by discussing the important borderline case $s = 1/2$.

**Theorem 4.17.** For any $\sigma > 0$, $T_\sigma$ is bounded from $(L^2(\sigma))\delta$ to $L^2(\sigma+\delta)$ with $\delta = \sigma/(1 + \sigma) > 0$.

**Proof.** By duality we need to prove that for all $f_1, \ldots, f_5, g \in L^2$ with $\|f_k\|_{L^2} = \|g\|_{L^2} = 1$, we have,

$$\langle T_\sigma (x)^{-\sigma} f_1, \ldots, (x)^{-\sigma} f_5, (x)^{\sigma+\delta} g \rangle_{L^2} \lesssim 1.$$

Unpacking this using (4.19), we see that this is the same as

$$\int_{-1}^{1} \int_{\mathbb{R}^3} K(x, y, z, \lambda) f_1(z - y + x) f_2(\lambda z + x) f_3(\lambda y - y + x)$$

$$(4.45) \quad \big| T_4(\lambda z - y + x) T_5(z + \lambda y - y + x) \big| dx dy dz d\lambda \lesssim 1,$$

where

$$K(x, y, z, \lambda) = \langle x \rangle^{\sigma+\delta} \langle z - y + x \rangle^\sigma \langle \lambda z + x \rangle^\sigma \langle \lambda y - y + x \rangle^\sigma \langle z + \lambda y - y + x \rangle^\sigma.$$

(The integration in $\lambda$ here is over $[-1, 1]$. The integral in $\lambda$ for $(-\infty, -1) \cup (1, \infty)$ can be transformed into this integral by the change of variables $\lambda \mapsto 1/\lambda$.)

The overall strategy is to identify a large set on which $K$ is bounded, where controlling the integral is easy, and use a dyadic decomposition and finer bounds on $T_\sigma$ to control the integral on the set where $K$ is not bounded. On the set where $K$ is bounded the boundedness property,

$$\langle T_\sigma (f_1, f_2, f_3, f_4, f_5, g) \rangle \lesssim \left( \prod_{k=1}^{5} \|f_k\|_{L^2}^5 \right) \|g\|_{L^2}.$$

deals with the integral automatically. So we only need to worry about the set where $K$ is unbounded. On the unbounded piece the refined estimate,

$$\langle (T_\sigma |\lambda - a| < \epsilon (f_1, \ldots, f, g) \rangle \leq \epsilon \left( \prod_{k=1}^{5} \|f_k\|_{L^2}^5 \right) \|g\|_{L^2},$$

will be used to gain control. This refined bound is a clear consequence of representation (4.20).

Fix $\epsilon$ small. We observe first that if $\epsilon|x| \leq 1$ then $K \lesssim 1$. So we assume that $\epsilon|x| \geq 1$.

Now the relation,

$$|z - y + x|^2 + |\lambda z + x|^2 + |\lambda y - y + x|^2 = |\lambda z - y + x|^2 + |z + \lambda y - y + x|^2 + |x|^2,$$

gives that,

$$|x| \leq \max \{|z - y + x|, |\lambda z + x|, |\lambda y - y + x|\},$$

and hence,

$$K \leq \langle x \rangle^\delta \langle \lambda z - y + x \rangle^\sigma \langle z + \lambda y - y + x \rangle^\sigma.$$

Now if,

$$|\lambda z - y + x| \geq \epsilon|x| \quad \text{or} \quad |z + \lambda y - y + x| \geq \epsilon|x|,$$
then we automatically get $K \lesssim 1$ as $\delta \leq \sigma$. Hence we assume that,

$$|z - y + x| \leq \epsilon|x| \quad \text{and} \quad |z + \lambda y - y + x| \leq \epsilon|x|.$$  

We now make some observations about how the sizes of $|y|$ and $|z|$ affect $K$. There are four cases.

1. First, suppose $|z|$ is large or comparable to $|x|$, so $|z| \geq 2\epsilon|x|$. Then,

$$2\epsilon|x| \leq |z| \leq |\lambda y - y + x| + |z + \lambda y - y + x| \leq |\lambda y - y + x| + \epsilon|x|,$$

and hence $\epsilon|x| \leq |\lambda y - y + x|$.

2. Next, suppose $|y|$ is large or comparable to $|x|$, so $|y| \geq 2\epsilon|x|$. Then,

$$2\epsilon|x| \leq |y| \leq |\lambda z + x| + |\lambda z - y + x| \leq |\lambda y - y + x| + \epsilon|x|,$$

and hence $\epsilon|x| \leq |\lambda z + x|$.

3. Next, suppose $|z|$ is small compared to $|x|$, so $|z| \leq 2\epsilon|x|$. Then,

$$|\lambda z + x| \geq |x| - |\lambda z| \geq |x| - |z| \geq (1 - 2\epsilon)|x|,$$

and so (by the smallness of $\epsilon$) we have $\epsilon|x| \leq |\lambda z + x|$.

4. Finally, suppose $|y|$ is small compared to $|y|$, so $|y| \leq 2\epsilon|x|$. Then,

$$|\lambda y - y + x| \geq |x| - (|\lambda| - 1)|y| \geq |x| - 2|y| \geq (1 - 4\epsilon)|x|,$$

and so (by the smallness of $\epsilon$) we have $\epsilon|x| \leq |\lambda y - y + x|$.

From these observations we see that if $|z|$ and $|y|$ are both large we get,

$$\langle x \rangle^{\sigma + \delta} \lesssim \langle \lambda z + x \rangle^{\sigma} (\lambda y - y + x)^{\sigma},$$

and hence $K \lesssim 1$. If $|z|$ and $|y|$ are both small then we have the same bound on $\langle x \rangle$ and the same conclusion.

There are thus two regimes to consider: when $|y|$ is large and $|z|$ small, and when $|z|$ is large and $|y|$ small. For these regimes we will use a dyadic decomposition:

$$\langle x \rangle \sim 2^j, \quad \langle \lambda z - y + x \rangle \sim 2^k \quad \text{and} \quad \langle z + \lambda y - y + x \rangle \sim 2^l.$$

**Regime One:** $|y|$ large, $|z|$ small.

From the observations above we have that $|\lambda z + x| \gtrsim \epsilon 2^j$. We have

$$|z - y + x| \geq |\lambda y| - |z + \lambda y - y + x| \geq \frac{\epsilon}{2} |\lambda x| \gtrsim |\lambda| 2^j$$

if we assume in addition that $|\lambda| 2^j \gtrsim \epsilon |\lambda| |x| / 2 \geq |z + \lambda y - y + x| = 2^j$. Under this assumption we have,

$$K \lesssim \frac{(2^j)^{\sigma + \delta}}{(2^j)^{(|\lambda| 2^j) / (2^j)}} \lesssim 2^{j(\delta - \sigma) / 2} |\lambda|^{-\sigma},$$

which is bounded if $2^{j(\delta / \sigma - 1) - l} \lesssim |\lambda|$. Hence in Regime One, $K$ is bounded unless,

$$|\lambda| \leq \max \left\{ 2^{j - l}, 2^{j(\delta / \sigma - 1) - l} \right\} =: \alpha.$$

By the bound,

$$K \lesssim \frac{2^j |\lambda|^{\sigma + \delta}}{2^j |\lambda|^{\sigma} 2^j |\lambda|^{\sigma}} = \frac{2^j \alpha}{2^j \alpha},$$

we then have

$$T_0 \text{ on Regime One set} \lesssim \sum_{\epsilon 2^j = 1} \sum_{\epsilon 2^l = 2^k} \sum_{\epsilon 2^l \geq 2^j} \frac{2^j}{2^j \alpha} \left( T_0 |\lambda| \leq \alpha (f_1, f_2, f_3, f_4, f_5, g) \right) \lesssim \sum_{\epsilon 2^j = 1} \sum_{\epsilon 2^l = 2^k} \sum_{\epsilon 2^l \geq 2^j} \frac{2^j}{2^j \alpha} \max \left\{ 2^{j - l}, 2^{j(\delta / \sigma - 1) - l} \right\} \lesssim 1.$$

**Regime Two:** $|z|$ large, $|y|$ small.

From the observations above we have that $|\lambda y - y + x| \gtrsim \epsilon 2^j$.

We have,

$$|z - y + x| \geq |(1 - \lambda)z| - |\lambda z - y + x| \geq \frac{\epsilon}{2} |(1 - \lambda)z| \gtrsim |1 - \lambda| 2^j,$$
if we assume in addition that $|1 - \lambda| 2^j \lesssim \epsilon |1 - \lambda| |x|/2 \geq |\lambda z - y + x| = 2^k$. Under this assumptions we have,

$$
K \lesssim \frac{(2^j)^{\sigma + \delta}}{(2^k)(|1 - \lambda|2^j)(2^i)} = 2^{j(\delta - \sigma)} 2^{-k\sigma} |1 - \lambda|^{-\sigma},
$$

which is bounded if $2^{j(\delta/\sigma - 1)} 2^{-k} \lesssim |1 - \lambda|$. Hence in Regime Two, $K$ is bounded unless,

$$
|1 - \lambda| \leq \max \left\{ 2^{k-j}, 2^{j(\delta/\sigma - 1)} 2^{-k} \right\} =: \alpha.
$$

By the bound,

$$
K \lesssim \frac{|2^j|^{\sigma + \delta}}{|2^k|^{\sigma} |2^i|^{\sigma} |2^j|^{\sigma}} = \frac{2^{j\delta}}{2^{k\sigma} 2^{i\sigma}},
$$

we then have

$$
\mathcal{T}_6 \text{ on Regime Two set } \lesssim \sum_{\epsilon 2^j \geq 1} \sum_{\epsilon 2^i \geq 2^i} \sum_{\epsilon 2^k \geq 2^i} \frac{2^{j\delta}}{2^{k\sigma} 2^{i\sigma}} \langle \mathcal{T}_{1-\lambda} \leq \alpha (f, \ldots, f), g \rangle
\lesssim \sum_{\epsilon 2^j \geq 1} \sum_{\epsilon 2^i \geq 2^i} \sum_{\epsilon 2^k \geq 2^i} \frac{2^{j\delta}}{2^{k\sigma} 2^{i\sigma}} \max \left\{ 2^{k-j}, 2^{j(\delta/\sigma - 1)} 2^{-k} \right\} \lesssim 1.
$$

The bound (4.45) is thus established.

**Theorem 4.18.** For all $s > 1/2$, there is a constant $C$ such that,

$$(4.46) \quad \|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_{L^{\infty,s}} \leq C \prod_{k=1}^5 \|f_k\|_{L^{\infty,s}}$$

**Lemma 4.19.** Suppose that $v_1^2 + v_2^2 + v_3^2 = 1$. Then

$$(4.47) \quad \langle v_1 y_1 + v_2 y_2 + v_3 y_3 \rangle \leq \sqrt{2} \left[ |v_1| \langle y_1 \rangle + |v_2| \langle y_2 \rangle + |v_3| \langle y_3 \rangle \right].$$

**Proof.** We have

$$
1 + (v_1 y_1 + v_2 y_2 + v_3 y_3)^2 \leq 2 \left( 1 + v_1^2 y_1^2 + v_2^2 y_2^2 + v_3^2 y_3^2 \right)
\leq 2 \left( v_1^2 (1 + y_1^2) + v_2^2 (1 + y_2^2) + v_3^2 (1 + y_3^2) \right)
\leq 2 \left( v_1^2 (1 + y_1^2)^{1/2} + v_2^2 (1 + y_2^2)^{1/2} + v_3^2 (1 + y_3^2)^{1/2} \right)^2.
$$

Taking square roots then gives (4.47).

**Proof of Theorem 4.18.** We may assume by rescaling that $\|f_k\|_{L^{\infty,s}} = 1$, which means $|f_k(t)| \leq \langle t \rangle^{-s}$. Set $y_k = (A(\lambda) x)_k$. Then, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$
\|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_{L^{\infty,s}} \leq \sup_{x_1 \in \mathbb{R}} \langle x_1 \rangle^s \mathcal{T}_6 \left( \langle t \rangle^{-s}, \langle t \rangle^{-s}, \langle t \rangle^{-s}, \langle t \rangle^{-s}, \langle t \rangle^{-s} \right) (x_1)
\leq \sup_{x_1 \in \mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{1}{\lambda^2 - \lambda + 1} \langle y_1 \rangle^s \langle y_2 \rangle^s \langle y_3 \rangle^s \langle x_2 \rangle^s \langle x_3 \rangle^s dx_2 dx_3 d\lambda.
$$

We will show that the integral of the Japanese bracket terms over $x_2$ and $x_3$ can be bounded by an absolute constant independent of $\lambda$. Because $\lambda^2 - \lambda + 1$ is integrable over $\mathbb{R}$, this will prove the bound.

By Cauchy–Schwarz, we have

$$
\int_{\mathbb{R}^2} \langle y_1 \rangle^s \langle y_2 \rangle^s \langle y_3 \rangle^s \langle x_2 \rangle^s \langle x_3 \rangle^s dx_2 dx_3 d\lambda 
\leq \left( \int_{\mathbb{R}^2} \langle y_1 \rangle^{2s} \langle y_2 \rangle^{2s} \langle y_3 \rangle^{2s} dx_2 dx_3 \right)^{1/2} \left( \int_{\mathbb{R}^2} \langle x_3 \rangle^{2s} dx_2 dx_3 \right)^{1/2}.
$$

The second integral here splits as $\int_{\mathbb{R}} \langle x_2 \rangle^{-2s} dx_2 \int_{\mathbb{R}} \langle x_3 \rangle^{-2s} dx_3$, and is thus finite as $s > 1/2$. 


To bound the first integral we must use some structure of $A(\lambda)$, which is given by,

$$A(\lambda) = \frac{1}{\lambda^2 - \lambda + 1} \begin{pmatrix} \lambda & 1 - \lambda & \lambda^2 - \lambda \\ \lambda^2 - \lambda & \lambda & 1 - \lambda \\ 1 - \lambda & \lambda^2 - \lambda & \lambda \end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$ 

First we observe that the matrix has the following property. If we fix a row $k$ and column $j$, then the determinant of the matrix obtained by deleting row $k$ and column $j$ is precisely $a_{kj}$ – the element in row $k$ and column $j$. This means that

$$a_{11} = \frac{\lambda}{\lambda^2 - \lambda + 1} = \det \left[ \frac{1}{\lambda^2 - \lambda + 1} \begin{pmatrix} \lambda & 1 - \lambda \\ \lambda^2 - \lambda & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix},$$

with similar formulas for $a_{21}$ and $a_{31}$.

Next, because $A$ is an isometry, the inverse matrix is just the transpose. Since $x = A^{-1}y$ we have the formula,

$$x_1 = a_{11}y_1 + a_{21}y_2 + a_{31}y_3.$$

$A$ is an isometry, so $a_{11}^2 + a_{21}^2 + a_{31}^2 = 1$. We can therefore use (4.47), from the previous lemma, raised to to the power $s$; it reads,

$$|x_1|^{2s} \lesssim |a_{11}|^{2s}|y_1|^{2s} + |a_{21}|^{2s}|y_2|^{2s} + |a_{31}|^{2s}|y_3|^{2s}.$$

Applying this to bound the first integral in (4.48), we then have,

$$\int_{\mathbb{R}^2} |\langle x_1 \rangle|^{2s} \frac{d\phi_1}{d\gamma_1} \frac{d\phi_2}{d\gamma_2} \frac{d\phi_3}{d\gamma_3} \lesssim \int_{\mathbb{R}^2} |a_{11}|^{2s} |y_1|^{2s} |\phi_1|^{2s} |\phi_2|^{2s} |\phi_3|^{2s} d\phi_2 d\phi_3 + \int_{\mathbb{R}^2} |a_{21}|^{2s} |y_2|^{2s} |\phi_1|^{2s} |\phi_2|^{2s} |\phi_3|^{2s} d\phi_2 d\phi_3 + \int_{\mathbb{R}^2} |a_{31}|^{2s} |y_3|^{2s} |\phi_1|^{2s} |\phi_2|^{2s} |\phi_3|^{2s} d\phi_2 d\phi_3.$$

We will show how the first integral may be bounded; the other two are bounded by an identical argument. We perform the change of variables $z_2 = y_2 = (Ax)_2$ and $z_3 = y_3 = (Ax)_3$. Expressed as a matrix, this change of variables is,

$$\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}.$$

The determinant of this change of variables is, by (4.49), simply $|a_{11}|$. Therefore, using that $|a_{11}| \leq 1$ because $A$ is an isometry,

$$\int_{\mathbb{R}^2} |a_{11}|^{2s} |y_1|^{2s} |\phi_1|^{2s} |\phi_2|^{2s} |\phi_3|^{2s} d\phi_2 d\phi_3 = \int_{\mathbb{R}^2} |a_{11}|^{2s-1} |x_1|^{2s} |\phi_2|^{2s} |\phi_3|^{2s} d\phi_2 d\phi_3 \leq \int_{\mathbb{R}} |x_1|^{2s} d\phi_1 \int_{\mathbb{R}} |x_2|^{2s} d\phi_2 \int_{\mathbb{R}} |x_3|^{2s} d\phi_3,$$

and the right hand side is finite because $s > 1/2$. \hfill \Box

**Proposition 4.20.** If $s < 1/2$, there is no constant $C$ such that $\|T_0(f, f, f, f, f)\|_{L^{\infty,s}} \leq C\|f\|_{L^{\infty,s}}^6$ for all functions $f \in L^{\infty,s}$.\n
The proof of the proposition involves a standard scaling argument, which we omit. The index $s = 1/2$, which is the borderline between Theorem 4.18 and Proposition 4.20, is especially relevant because the space $L^{\infty,1/2}$ is a critical space for the equation $iu_t = T_0(u, u, u, u, u)$. That is, both the equation and the space are invariant under the scaling $f(x) \mapsto \lambda^{1/2} f(\lambda x)$. By analogy with the cubic continuous resonant equation in dimension two, it seems reasonable to conjecture that the operator $T_0$ is bounded from $(L^{\infty,1/2})^6$ to $L^{\infty,1/2}$. In fact, this is equivalent to the statement that

$$\sup_{x \in \mathbb{R}} T_0 \left( \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}} \right) (x) \sqrt{x} < \infty,$$

which by scaling is equivalent to,

$$w = T_0 \left( \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}} \right) (1) < \infty,$$
and again by scaling, this is equivalent to

\[ \frac{w}{\sqrt{x}} = T_0 \left( \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}} \right)(x), \]

for some $|w| < \infty$. In all, boundedness from $(L^{\infty,1/2})^5$ to $L^{\infty,1/2}$ is equivalent to $1/\sqrt{x}$ being a stationary wave system. The proof, unfortunately, does not extend to the present situation in a manageable way.

As in the case of quintic resonant system, Hamilton’s equation is

\[ iu_t = T_3(u, u, u) \]

by,

\[ T_3 (f_1, f_2, f_3, f_4) \]

The functional has a large number of permutation symmetries,

\[ E_4 (f_1, f_2, f_3, f_4) = E_4 (f_2, f_1, f_3, f_4) = E_4 (f_1, f_2, f_3, f_4) = E_4 (f_3, f_4, f_1, f_2). \]

As in the case of quintic resonant system, Hamilton’s equation is $i u_t = T_4 (u, u, u)$ where $T_4$ is defined by,

\[ (T_4 (f_1, f_2, f_3), g) = 4 E_4 (f_1, f_2, f_3, g). \]

By an identical computation to the derivation of (4.7), we find that the operator $T_4$ is given explicitly by,

\[ T_4 (f_1, f_2, f_3) (x) = \frac{8}{\pi} \int_{-\pi/4}^{\pi/4} e^{-i t} \left[ (e^{i t} f_1) (e^{i t} f_2) \right] (x) dt. \]

This shows that the flow corresponding to the Hamiltonian $H_4$ is precisely the resonant equation (2.9) in the case $k = 1$. As discussed in the introduction, it was shown in [21] that this system is also the modified scattering limit of the NLS equation (1.6).

5.1. Representations of the Hamiltonian and the flow operator. As for the quintic case, we devote a significant amount of work to determining alternative representations of $E_4$, $H_4$ and $T_4$. In contrast to the quintic case, we do not have representations for $H_4$ of the form,

\[ \int_{\mathbb{R}^4} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) \delta_{y_1 + y_2 = y_3 + y_4} \delta_{y_1^2 + y_2^2 + y_3^2 + y_4^2} dy \]

These representations are inconsistent with the scaling of the inequality $H_4 (f) \leq (1/\sqrt{8\pi}) \| f \|_{L^2}$ which we prove in Theorem 5.7.

**Theorem 5.1.** There holds the representations,

\[ E_4 (f_1, f_2, f_3, f_4) = \frac{1}{2 \pi^2} \int_{\mathbb{R}^4} e^{-\frac{i}{2} \left[ (\lambda v_2)^2 + v_1^2 \right]} f_1 (\lambda v_1 + v_3) f_2 (v_2 + v_4) f_3 (\lambda v_1 + v_2 + v_4) \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \lambda, \]

\[ T_4 (f_1, f_2, f_3) (x) = \frac{2}{\pi^2} \int_{\mathbb{R}^4} e^{-\frac{i}{2} \left[ (\lambda v_2)^2 + v_1^2 \right]} f_1 (\lambda v_1 + x) f_2 (v_2 + x) f_3 (\lambda v_1 + v_2 + x) \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \lambda, \]

To prove this theorem we need a lemma.
Proof. It is clear that \( \psi \) is in \( L^2 \). We will calculate the Fourier transform of \( \zeta(\xi) \) and find that it equals \( \psi \). The lemma then follows from Fourier inversion and the fact that \( \psi \) is even.

We have,

\[
\hat{\zeta}(x) = \frac{1}{2\pi} \int e^{ix\xi} \int \frac{1}{|v|} e^{-\frac{1}{2}((\xi/v)^2 + v^2)} dv d\xi
\]

\[
= \frac{1}{2\pi} \int \frac{1}{|v|} e^{-\frac{1}{2}v^2} \int e^{i\xi v} e^{-\frac{1}{2}(\xi/v)^2} d\xi dv = \frac{1}{2\pi} \int e^{-\frac{1}{2}v^2} e^{-\frac{1}{2}v^2 x^2} dv,
\]

where in the last equality we used the explicit Fourier transform of the Gaussian \( e^{-ax^2} \) with \( a = 1/(2v^2) \). In this last integral we perform the change of variables \( u = v(1 + x^2)^{-1/2} \), which gives,

\[
\hat{\zeta}(x) = \frac{1}{(1 + x^2)^{1/2}} \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}u^2} du = \frac{1}{(1 + x^2)^{1/2}} = \psi(x),
\]

which is what we wanted to prove. \( \square \)

Proof of Theorem 5.1. We evaluate (5.2) using the Mehler formula (2.2), which reads,

\[
e^{itH} f_k(x) = \frac{1}{\sqrt{2\pi}|\sin(2t)|^{1/2}} \int e^{-i \frac{x^2}{2} + \frac{1}{2} \frac{t^2}{\sin(2t)} \frac{\cos(2t) - Ex}{2\pi}} f_k(y) dy.
\]

For notational convenience, let \( \Lambda(x, t) = (e^{itH} f_1)(e^{itH} f_2)(e^{itH} f_3)(e^{itH} f_4) \) be the integrand in (5.2). Using the Mehler formula, we have

\[
\Lambda(x, t) = \frac{1}{4\pi^2 |\sin(2t)|^2} \int_{\mathbb{R}^4} e^{-i \frac{\Omega \cos(2t)}{2\pi \sin(2t)} e^{-i \frac{(x_1 + x_2 - y_3 - y_4)^2}{2\pi \sin(2t)}} f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4) dy_1 dy_2 dy_3 dy_4
\]

where \( \Omega = y_1^2 + y_2^2 - y_3^2 - y_4^2 \). Changing variables \( w(y_3) = -y_1 - y_2 + y_3 + y_4 \) and integrating over \( x \) yields,

\[
\int \Lambda(x, t) dx
\]

\[
= \frac{1}{4\pi^2 |\sin(2t)|^2} \int_{\mathbb{R}^4} e^{-i \frac{\Omega \cos(2t)}{2\pi \sin(2t)} e^{-i \frac{w^2}{2\pi \sin(2t)}} f_1(y_1) f_2(y_2) f_3(w + y_1 + y_2 - y_4) f_4(y_4) dw dy_1 dy_2 dy_3 dy_4
\]

\[
= \frac{1}{2\pi |\sin(2t)|} \int_{\mathbb{R}^3} e^{-i \frac{\Omega \cos(2t)}{2\pi \sin(2t)}} f_1(y_1) f_2(y_2) f_3(y_1 + y_2 - y_4) f_4(y_4) dy_2 dy_3 dy_4,
\]

where to get the second equality we used the Fourier inversion formula (1.18) with with \( a = 1 / \sin(2t) \).

We now integrate \( t \) on the interval \([-\pi/4, \pi/4] \) and then perform the change of variables \( u = -\cos(2t) / \sin(2t) \). This change of variables bijectively maps \((-\pi/4, 0) \cup (0, \pi/4)\) to \((-\infty, +\infty)\) and satisfies \( du = 2dt / \sin^2(2t) \). Moreover,

\[
u^2 = \frac{\cos^2(2t)}{\sin^2(2t)} = \frac{1}{\sin^2(2t)} - 1,
\]

which gives \( \sin(2t) = (u^2 + 1)^{-1/2} \). Using these, we find,

\[
\int_{-\pi/4}^{\pi/4} \int \Lambda(x, t) dx dt
\]

\[
= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( \int e^{-i \frac{\Omega u}{2} \frac{1}{(1 + u^2)^{1/2}}} du \right) f_1(y_1) f_2(y_2) f_3(y_1 + y_2 - y_4) f_4(y_4) dy_2 dy_3 dy_4.
\]
We notice that the term inside round parentheses is the Fourier transform of $\sqrt{2\pi}(1+u^2)^{-1/2}$ evaluated at the point $-\Omega/2$. Therefore, by Lemma 5.2, we have,

$$\int_{-\pi/4}^{\pi/4} \Lambda(x, t) dx dt = \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( \int_{|v|} e^{-\frac{i}{2}[(\Omega/2v^2 + v_1^2)]} dv_1 \right) f_1(-y_2 + y_3 + y_4) f_2(y_2) f_3(y_3) f_4(y_4) dy_2 dy_3 dy_4$$

At this point $\Omega/2 = [(y_1 + y_2)^2 - (y_1 + y_2 - y_4)^2 - (y_4)^2]/2 = (y_1 - y_4)(y_2 - y_4)$. For fixed $v_1$, we perform the linear change of variables,

$$\begin{pmatrix} \lambda \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (y_1 - y_4)/v_1 \\ y_2 - y_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1/v_1 & 0 & -1/v_1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ v_2 \\ v_3 \end{pmatrix},$$

which has determinant $1/|v_1|$. The inverse is given by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1 + v_3 \\ v_2 + v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ v_2 \\ v_3 \end{pmatrix},$$

and we note specifically that $\Omega/2v_1 = [(y_1 - y_4)/v_1](y_2 - y_4) = \lambda v_2$. This gives

$$\int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} \Lambda(x, t) dx dt = \frac{1}{4\pi} \int_{\mathbb{R}^4} e^{-\frac{i}{2}[(\lambda v_2)^2 + v_1^2]} f_1(\lambda v_1 + v_3) f_2(v_2 + v_3) f_3(\lambda v_1 + v_2 + v_3) f_4(v_3) dv_1 dv_2 dv_3 d\lambda,$$

which is formula (5.6).

To get (5.7), we simply use the relation $\langle T_4(f_1, f_2, f_3, g) = 4E_4(f_1, f_2, f_3, f_4)$. □

**Theorem 5.3.** Let $G(x) = e^{-\frac{1}{2}x^2}$. There holds the representations,

$$E_4(f_1, f_2, f_3, f_4) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} E_B(\lambda) (G, f_1, f_2, G, f_3, f_4) d\lambda$$

$$T_4(f_1, f_2, f_3) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} T_B(\lambda) (G, f_1, f_2, G, f_3) d\lambda$$

where for every $\lambda$, $B(\lambda)$ is an isometry and $B(\lambda)(0, 1, 1) = (0, 1, 1)$.

**Proof.** We observe that

$$(\lambda v_2)^2 + (v_1)^2 = \left( \frac{\lambda v_2 - v_1}{\sqrt{2}} \right)^2 + \left( \frac{\lambda v_2 + v_1}{\sqrt{2}} \right)^2$$

which gives

$$e^{-\frac{i}{2}[(\lambda v_2)^2 + (v_1)^2]} = G \left( \frac{\lambda v_2 - v_1}{\sqrt{2}} \right) G \left( \frac{\lambda v_2 + v_1}{\sqrt{2}} \right).$$

We substitute this expression into (5.6). Using the fact that $G(x) = G(x)$, this gives,

$$E_4(f_1, f_2, f_3, f_4) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^4} G \left( \frac{\lambda v_2 + v_3}{\sqrt{2}} \right) f(\lambda v_1 + v_3) f(v_2 + v_3) G \left( \frac{\lambda v_2 - v_1}{\sqrt{2}} \right) f(\lambda v_1 + v_2 + v_3) dv_1 dv_2 dv_3.$$

By looking at the arguments of the functions in the integrand, we are led to define the matrices $C(\lambda)$ and $D(\lambda)$ by,

$$(v_1 \ v_2 \ v_3) = \begin{pmatrix} 1/\sqrt{2} & \lambda/\sqrt{2} & 0 \\ \lambda & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} (v_1 \ v_2 \ v_3) = \begin{pmatrix} (\lambda v_2 + v_1)/\sqrt{2} \\ \lambda v_1 + v_3 \\ v_2 + v_3 \end{pmatrix}$$

and,

$$(v_1 \ v_2 \ v_3) = \begin{pmatrix} -1/\sqrt{2} & \lambda/\sqrt{2} & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (v_1 \ v_2 \ v_3) = \begin{pmatrix} (\lambda v_2 - v_1)/\sqrt{2} \\ \lambda v_1 + v_2 + v_3 \\ v_3 \end{pmatrix}.$$
We perform the change of variables \( w = D(\lambda)v \). We determine that \( \det D(\lambda) = (1 + \lambda^2)/\sqrt{2} > 0 \), and so \( v = D(\lambda)^{-1}w \) is well defined. Set \( B(\lambda) = C(\lambda)D(\lambda)^{-1} \). Performing the change of variables then yields the expression (5.8). The equation for \( T_4 \) follows from (5.4). A calculation reveals that \( B(\lambda) \) is given explicitly by

\[
B(\lambda) = C(\lambda)D(\lambda)^{-1} = \frac{1}{1 + \lambda^2} \begin{pmatrix} -1 + \lambda^2 & \lambda\sqrt{2} & -\lambda\sqrt{2} \\ -\lambda\sqrt{2} & \lambda^2 & 1 \\ \lambda\sqrt{2} & 1 & \lambda^2 \end{pmatrix}.
\]

We finally verify the two properties of \( B(\lambda) \).

1. The relationship

\[
\left( \frac{\lambda v_2 + v_1}{\sqrt{2}} \right)^2 + (\lambda v_1 + v_3)^2 + (v_2 + v_3)^2 = \left( \frac{\lambda v_2 - v_1}{\sqrt{2}} \right)^2 + (\lambda v_1 + v_2 + v_3)^2 + (v_3)^2
\]

means that for all \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \) we have \( |C(\lambda)v|^2 = |D(\lambda)v|^2 \). Replacing \( v \) with \( D(\lambda)^{-1}v \) gives \( |B(\lambda)v| = |v| \), so that \( B(\lambda) \) is an isometry.

2. Set \( e = (0, 1, 1) \). The relationship among the arguments of the functions \( f_k \),

\[
(\lambda v_1 + v_3) + (v_2 + v_3) = (\lambda v_1 + v_2 + v_3) + (v_3),
\]

means that for all \( v \in \mathbb{R}^3 \), \( \langle C(\lambda)v, e \rangle = \langle D(\lambda)v, e \rangle \). Replacing \( v \) by \( D(\lambda)^{-1}v \), we have \( \langle v, e \rangle = \langle B(\lambda)v, e \rangle = \langle v, B(\lambda)^{-1}e \rangle \), which means \( e = B(\lambda)^{-1}e \) and hence \( B(\lambda)e = e \). □

**Theorem 5.4.** There holds the representations,

\[
E_4(f_1, f_2, f_3, f_4) = \frac{1}{2\sqrt{2}\pi^2} \int_0^{2\pi} E_S(\theta)(G, f_1, f_2, G, f_3, f_4) d\theta,
\]

\[
T_4(f_1, f_2, f_3)(x) = \frac{\sqrt{2}}{\pi^2} \int_0^{2\pi} T_S(\theta)(G, f_1, f_2, G, f_3)(x) d\theta,
\]

where \( S(\theta) \) is the rotation of \( \mathbb{R}^3 \) by \( \theta \) radians about the axis \((0, 1, 1)\).

**Proof.** Because the matrix \( B(\lambda) \) is an isometry, \( \det(B(\lambda)) = +1 \), and \( B(\lambda)(0, 1, 1) = (0, 1, 1) \), it must, in fact, be a rotation about the axis \((0, 1, 1)\).

For any rotation \( A \) of \( \mathbb{R}^3 \), the angle of rotation \( \theta \) satisfies, \( 2\cos(\theta) + 1 = \text{Trace}(A) \). In the present case, this means,

\[
\cos(\theta) = \phi(\lambda) := \frac{1}{2} (\text{Trace}(B(\lambda)) - 1) = \frac{1}{2} \left( \frac{1 - 3\lambda^2}{1 + \lambda^2} - 1 \right).
\]

We carefully perform the change of variables \( \lambda \mapsto \theta \) in (5.8).

We find that \( \phi(0) = 1 \), that \( \phi \) is increasing on \((-\infty, 0]\), decreasing on \([0, +\infty)\), and that as \( \lambda \to \pm\infty \), \( \phi(\lambda) \to -1 \). By setting \( \lambda = 1 \) in (5.10), we find that \( \sin(\theta) = -1 < 0 \), and hence \( \theta = -\pi/2 \in [\pi, 2\pi] \).

By setting \( \lambda = -1 \) in (5.10), we find that \( \sin(\theta) = 1 > 0 \), and hence \( \theta = \pi/2 \in [0, \pi] \). From these considerations and continuity, we infer that under \( \lambda \mapsto \theta \), \( \mathbb{R} \) is bijectively mapped to \((0, 2\pi]\).

To perform the change of variables, we need to find its determinant. It is given by,

\[
\frac{d\theta}{d\lambda} = \left| \frac{d}{d\lambda} \arccos \left( \frac{1}{2} \begin{pmatrix} -1 + 3\lambda^2 & -1 \\ \lambda^2 + 1 & -1 \end{pmatrix} \right) \right| = \left| \frac{d}{d\lambda} \arccos \left( \frac{\lambda^2 - 1}{\lambda^2 + 1} \right) \right| = \left| \frac{d}{d\lambda} \arctan \left( \frac{2\lambda}{\lambda^2 + 1} \right) \right| = 2 \left| \frac{d}{d\lambda} \arctan(\lambda) \right| = \frac{2}{1 + \lambda^2}.
\]

Formula (5.11) then follows. □

Formula (5.11) is very similar to the formula (4.26) derived in Section 4,

\[
E_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_R(\theta)(f_1, f_2, f_3, f_4, f_5, f_6) d\theta,
\]
where $R(\theta)$ is the rotation of $\mathbb{R}^3$ by $\theta$ radians about the axis $(1, 1, 1)$. In both cases the functionals $E_6$ and $E_4$ are expressed in terms of a family of rotations about a fixed axis. The specific axis in both cases is basically determined by the fact that both functionals are translation invariant.

The presence of $G$ in (5.11) has concrete consequences for the symmetry structure of $E_4$. While the functionals $E_{S(\theta)}$ are all invariant under $L^2$ scaling, for example, this is not inherited by $E_4$ because the scaling transformation also affects the $G$ terms, but these must be fixed. Similarly, the functional $E_4$ is not invariant under the actions $f_k \mapsto e^{i\lambda x^2}f_k$ and $f_k \mapsto e^{i\lambda H}f_k$. (However we will see that the functional is invariant under the action $f_k \mapsto e^{i\lambda H}f_k$ because $G$ is a fixed point for this.) Finally, we will also see that the presence of the $G$ terms has consequences for the set of saturating functions of the $L^2$ bound for $E_4$.

5.2. Symmetries of the Hamiltonian and conserved quantities of the flow.

**Theorem 5.5.** The function $E_4(f_1, f_2, f_3, f_4)$ is invariant under the following actions:

(i) Fourier transform, $f_k \mapsto \hat{f}_k$.

(ii) Modulation, $f_k \mapsto e^{i\lambda}f_k$.

(iii) Linear modulation, $f_k \mapsto e^{i\lambda}f_k$.

(iv) Translation, $f_k \mapsto f_k(\cdot + \lambda)$.

(v) Schrödinger with harmonic trapping group, $f_k \mapsto e^{i\lambda H}f_k$.

**Proof.** Because $S(\theta)$ is an isometry for all $\theta$, items (i) through (v) of Theorem 3.3 apply to the functional $E_{S(\theta)}$.

(i) Let $F$ denote the Fourier transform. Note that $F(G) = F(e^{-x^2/2}) = e^{-x^2/2} = G$. Therefore

$$E_4(\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4) = \frac{1}{\sqrt{8\pi^2}} \int_0^{2\pi} F\left(E_{S(\theta)}(G, \hat{f}_1, \hat{f}_2, G, \hat{f}_4)\right) d\theta$$

$$= \frac{1}{\sqrt{8\pi^2}} \int_0^{2\pi} E_{S(\theta)}(\hat{G}, f_1, f_2, \hat{G}, f_3, f_4) d\theta = E_4(f_1, f_2, f_3, f_4).$$

(ii) This is an easy consequence of representation (5.6).

(iii) This again follows from representation (5.6).

(iv) This follows from (iii), along with the Fourier transform symmetry in (i).

(v) Recall that $G(x) = e^{-x^2/2}$ is (a constant multiple of) the first Hermite function and hence satisfies $e^{iH}G(x) = e^{iH}G(x)$. This then gives, for each $\theta$ and $s$,

$$E_{S(\theta)}(G, e^{i\lambda}f_1, e^{i\lambda}f_2, G, e^{i\lambda}f_3, e^{i\lambda}f_4) = E_{S(\theta)}(e^{-i\lambda}G, f_1, f_2, e^{-i\lambda}G, f_3, f_4)$$

$$= E_{S(\theta)}(e^{-i\lambda}G, f_1, f_2, e^{-i\lambda}G, f_3, f_4)$$

$$= E_{S(\theta)}(G, e^{i\lambda}f_1, e^{i\lambda}f_2, e^{i\lambda}f_3, e^{i\lambda}f_4)$$

This then gives,

$$E_4(e^{i\lambda}f_1, e^{i\lambda}f_2, e^{i\lambda}f_3, e^{i\lambda}f_4) = E_4(e^{i\lambda}f_1, e^{i\lambda}f_2, e^{i\lambda}f_3, e^{i\lambda}f_4) = E_4(f_1, f_2, f_3, f_4),$$

using (ii). \hfill \Box

**Corollary 5.6.** We have the following commutator equalities,

$$e^{i\lambda Q}T_4(f_1, f_2, f_3) = T_4(e^{i\lambda Q}f_1, e^{i\lambda Q}f_2, e^{i\lambda Q}f_3)$$

$$Q T_4(f_1, f_2, f_3) = T_4(Q f_1, f_2, f_3) + T_4(f_1, Q f_2, f_3) - T_4(f_1, f_2, Q f_3).$$

where $Q$ are the operators: $Q = 1, Q = x, Q = id/dx$, and $Q = H$.

The corollary follows immediately from Theorem 3.4. By Noether’s Theorem, Theorem 3.5, we determine four conserved quantities for the Hamiltonian flow corresponding to $H_4$. These are summarized in the following table.
This gives \( \alpha \) and hence \( \alpha \) isometry, the terms involving \( - (5.19) \) must necessarily be Gaussians by Theorem 3.12. To find which Gaussians are admissible, write, for almost all \( \alpha \) we necessarily have the set of saturating functions is all Gaussians of the form \( e^{(5.17)} \) with equality if and only if \( f(x) = e^{-\frac{1}{2}x^2 + \beta x} \) for some \( \beta \in \mathbb{C} \).

In particular there holds \( \mathcal{H}_4(f) \leq (1/\sqrt{2\pi})\|f\|_{L^2}^2 \) with equality if and only if \( f(x) = e^{-\frac{1}{2}x^2 + \beta x} \) for some \( \beta \in \mathbb{C} \).

The equality case here is a little different to the analogous result for \( \mathcal{E}_6 \) in Theorem 4.10. For \( \mathcal{E}_6 \), the set of saturating functions is all Gaussians of the form \( e^{-\alpha x^2 + \beta x} \) with \( \text{Re} \alpha > 0 \). In the case of \( \mathcal{E}_4 \), we necessarily have \( \alpha = 1/2 \).

**Proposition 5.7.** We have the following sharp bound,

\[
|\mathcal{E}_4(f_1, f_2, f_3, f_4)| \leq \frac{1}{\sqrt{2\pi}} \prod_{k=1}^4 \|f_k\|_{L^2},
\]

with equality if and only if the functions \( f_k \) are the same Gaussian \( f_k(x) = e^{-\frac{1}{2}x^2 + \beta x} \) for some \( \beta \in \mathbb{C} \).

We calculate \( \|G\|_{L^2}^2 = \int_{\mathbb{R}} (e^{-x^2/2})^2 dx = \sqrt{\pi} \), which yields the inequality.

To have equality, we must have

\[
|\mathcal{E}_4(f_1, f_2, f_3, f_4)| = \|G\|_{L^2}^2 \prod_{k=1}^4 \|f_k\|_{L^2},
\]

for almost all \( \theta \). Because \( S(\theta) \) is not a signed permutation almost everywhere, functions \( f_k \) satisfying (5.18) must necessarily be Gaussians by Theorem 3.12. To find which Gaussians are admissible, write,

\[
f_k(x) = e^{-\frac{1}{2}x^2 + \alpha_k x^2 + \beta_k x},
\]

for \( \alpha_k, \beta_k \in \mathbb{C} \). The equality condition from Theorem 3.12 reads, for all \( \theta \) and all \( x \in \mathbb{R}^3 \),

\[
G((S(\theta)x)_1)f_1((S(\theta)x)_2)f_2((S(\theta)x)_3) = G(x_1)f_3(x_2)f_4(x_3)
\]

We substitute the expressions for \( f_k \) in (5.19) and \( G(x) = e^{-x^2/2} \) into (5.20). Because \( S(\theta) \) is an isometry, the terms involving \(-x^2/2 \) will cancel, leaving,

\[
(\alpha_1 + \alpha_2) \sin^2(\theta)/2 + (\beta_1 + \beta_2) \sin(\theta)/\sqrt{2} = 0,
\]

and hence \( \alpha_1 = -\alpha_2 \) and \( \beta_1 = \beta_2 \). Setting \( x = S(-\theta)(1,0,0) \) similarly yields \( \alpha_3 = -\alpha_4 \) and \( \beta_3 = \beta_4 \).

Next set \( x = (1,0,0) \). Then \( S(\theta)x = (\cos(\theta), -\sin(\theta)/\sqrt{2}, \sin(\theta)/\sqrt{2}) \), which gives,

\[
(\alpha_1 + \alpha_2) \frac{\sin^2(\theta)}{2} + (\beta_1 - \beta_2) \frac{\sin(\theta)}{\sqrt{2}} = 0,
\]

5.3. Boundedness of the functional and wellposedness of Hamilton’s equation.

\[
\text{Symmetry of } \mathcal{H}_4 \quad \text{Conserved quantity} \quad \text{Operator commuting with } \mathcal{T}_4
\]

| \( f \mapsto e^{i\lambda} f \) | \( \int_{\mathbb{R}} |f(x)|^2 dx \) | 1 |
| \( f \mapsto e^{i\lambda x} f \) | \( \int_{\mathbb{R}} x|f(x)|^2 dx \) | \( x \) |
| \( f \mapsto f(+\lambda) \) | \( \text{Re} \int_{\mathbb{R}} f(x)\overline{f}(x) dx \) | \( d/dx \) |
| \( f \mapsto e^{i\lambda H} f \) | \( \int_{\mathbb{R}} |xf(x)|^2 + |f'(x)|^2 dx \) | \( H \) |
Finally set \( x = (1, 1, 1) = (1, 0, 0) + (0, 1, 1) \). Then \( S(\theta)x = (\cos(\theta), 1 - \sin(\theta)/\sqrt{2}, 1 + \sin(\theta)/\sqrt{2}) \), and,
\[
\alpha_1 \left( 1 - \frac{\sin(\theta)}{\sqrt{2}} \right)^2 - \alpha_1 \left( 1 + \frac{\sin(\theta)}{\sqrt{2}} \right)^2 = \alpha_3 + \alpha_4 = 0,
\]
expanding the left hand side we find that \( 2\alpha_1 \sin(\theta) = 0 \), and hence \( \alpha_1 = \alpha_2 = 0 \). Similarly by considering \( x = S(-\theta)(1,1,1) \) we find that \( \alpha_3 = \alpha_4 = 0 \).

In conclusion, the functions \( f_k \) in (5.19) satisfy the equality condition (5.20) if and only if \( \alpha_k = 0 \) and \( \beta_k = \beta_1 \) for all \( k \).

**Theorem 5.8.** We have the operator bound
\[
\|T_4(f_1, f_2, f_3)\|_X \leq C_X \prod_{k=1}^3 \|f_k\|_X,
\]
for the spaces
(i) \( X = L^2 \) with \( C_X = \sqrt{8/\pi} \).
(ii) \( X = L^{2,\sigma} \), for any \( \sigma \geq 0 \).
(iii) \( X = H^\sigma \), for any \( \sigma \geq 0 \).
(iv) \( X = L^{\infty,s} \), for any \( s > 1/2 \).
(v) \( X = L^{p,s} \), for any \( p \geq 2 \) and \( s > 1/2 - 1/p \).

**Proof.** The bounds (i) through (iii) follow from the representation (5.12) along with bounds on \( T_{S(\theta)} \) from Theorem 3.7, noting that in all cases \( \|G\|_X < \infty \).

For (iv), we need to show \( \sup_{x \in \mathbb{R}} |T_4((t)^{-s}, (t)^{-s}, (t)^{-s}) (x) (x)^{s}| < \infty \). As in the proof of Theorem 4.18, it is sufficient to show that,
\[
\sup_{x \in \mathbb{R}} T_{B(\lambda)} (e^{-t^2/2}, (t)^{-s}, (t)^{-s}, e^{-t^2/2}, (t)^{s}) (x) (x)^{-s} \leq C,
\]
for some \( C \) independent of \( \lambda \). We observe that we have \( e^{-t^2/2} \lesssim (t)^{-s} \), which means it is sufficient to show that,
\[
\sup_{x \in \mathbb{R}} T_{B(\lambda)} ((t)^{-s}, (t)^{-s}, (t)^{-s}, (t)^{-s}, (t)^{-s}) (x) (x)^{s} \leq C,
\]
for some \( C \) independent of \( \lambda \). The proof of this bound is identical to the proof of the analogous bound in Theorem 4.18.

Item (v) follows from interpolating between \( L^{2,\sigma} \) and \( L^{\infty,s} \).\( \square \)

**Theorem 5.9.** Consider the Cauchy problem,
\[
(5.22) \quad iu_t = T_4(u, u, u),
\]
which is Hamilton’s equation corresponding to \( \mathcal{H}_4 \) and the resonant equation (1.4) in the cubic case \( k = 1 \).

(i) The Cauchy problem (5.22) is locally wellposed in \( X \) for any of the spaces in the previous theorem.

(ii) The Cauchy problem (5.22) is globally wellposed in \( L^2 \)

**Proof.** Identical to that of Theorem 4.12. \( \square \)

5.4. **Analysis of the stationary waves.** As for the quintic resonant equation, Theorem 3.10 may be used to produce stationary solutions of the equation \( iu_t = T_4(u, u, u) \). In fact, if \( \phi_n \) is a Hermite function we have, recalling that \( G(x) = c_0 \phi_0 \),
\[
T_{S(\theta)} (G, \phi_n, G, \phi_n) = c_0^2 T_{S(\theta)} (\phi_0, \phi_n, \phi_0, \phi_n) = C \phi_n,
\]
for some constant \( C \), by Theorem 3.10. This immediately implies that
\[
T_4 (\phi_n, \phi_n, \phi_n) = \frac{1}{2\sqrt{2\pi}^2} \int_0^{2\pi} T_{S(\theta)} (G, \phi_n, G, \phi_n) d\theta = C \phi_n,
\]
and hence that $\phi_n$ is a stationary wave for the Hamiltonian flow of $H_4$. By taking the inner product with respect to $\phi_n$ and using $H_4(\phi_n) = \|e^{itH} \phi_n\|_{L_t^2 L_x^4} = \|\phi_n\|_{L_t^4}$, one finds that $C = \|\phi_n\|^4 / 4$.

By applying the symmetries of $H_4$, we find that all functions of the form,

\begin{equation}
\tag{5.23}
 ae^{ibx} \phi_n(x + c),
\end{equation}

are stationary waves for $a \in \mathbb{C}$ and $b, c \in \mathbb{R}$. The set of stationary waves we can construct for the cubic case is smaller than the set we can construct for the quintic case in (4.37), because the cubic equation has fewer symmetries.

5.4.1. Regularity of stationary waves: technical issues. All of the stationary waves constructed in the previous subsection are analytic and exponentially decaying in space. In the remainder of this section we prove that all stationary waves that are in $L^2$ are automatically analytic and decay in space like $e^{-\alpha x^2}$ for some $\alpha > 0$. This is analogous to Corollary 4.16 for the quintic resonant equation. Recall that the proof of that result relied on two ingredients: a refined multilinear Strichartz estimate (4.39) and a weight transfer property (4.38).

The weight transfer property was a direct result of the weight transfer lemma, Lemma 3.13, for the functionals $E_A$. In the present case we encounter a problem when trying to replicate this: when we try to transfer weight in the functional $E_A$ using Lemma 3.13, the weight also hits the Gaussians,

\begin{equation}
\tag{5.24}
 E_B(\lambda)(G, f_1, f_2, G, f_3, f_4 G, G, \mu, c, f_4) \leq E_B(\lambda)(G, \mu, e, f_1 G, G, f_2 G, G, f_3 G, f_4 G, f_4),
\end{equation}

and the right hand side here can’t be related back to $E_A$. To get around this, we observe that,

\begin{equation}
\tag{5.25}
 G(x) G, \mu, e(x) = e^{-\frac{1}{2}x^2} e^{x^2/(1 + \epsilon x^2)} \leq e^{-\frac{1}{2}x^2},
\end{equation}

which, if $\mu < 1/2$, is still decaying exponentially fast, and should be possible to handle in estimates.

Because of this consideration, we are led to define,

\begin{equation}
\tag{5.26}
 \mathcal{E}_4^\mu(\mu_1, f_2, f_3, f_4) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} E_B(\lambda) \left( e^{(-\frac{1}{2}+\mu) x^2}, f_1, f_2, e^{(-\frac{1}{2}+\mu) x^2}, f_3, f_4 \right) d\lambda,
\end{equation}

and we note that $\mathcal{E}_4^0 = \mathcal{E}_4$. We now proceed to develop the two ingredients for the stationary wave result, noting that both ingredients need to be developed for $\mathcal{E}_4^\mu$ and not just $\mathcal{E}_4$.

5.4.2. The weight transfer property.

**Lemma 5.10.** (i) If $\mu < \frac{1}{2}$ and functions $f_k$ are positive then there holds,

\begin{equation}
\tag{5.27}
 \mathcal{E}_4(\mu_1, f_2, f_3, f_4 G, \mu, c) \leq \mathcal{E}_4^\mu(\mu_1 G, \mu, e(x), f_2 G, G, \mu, e(x), f_3 G, G, \mu, e(x)), f_4).
\end{equation}

(ii) If $\mu < \frac{1}{2}$ then there holds the bound,

\begin{equation}
\tag{5.28}
 |\mathcal{E}_4^\mu(\mu_1, f_2, f_3, f_4)| \leq \sqrt{\frac{\pi}{8 \sqrt{1 - 2\mu}}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}.
\end{equation}

**Proof.** (i) This is immediate from the computations in (5.24) and (5.25).

(ii) Boundedness is proved in the usual way,

\begin{align*}
|\mathcal{E}_4^\mu(\mu_1, f_2, f_3, f_4)| & \leq \frac{1}{\sqrt{8\pi}} \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} \left\| e^{(-\frac{1}{2}+\mu) x^2} \right\|_{L^2} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2} d\lambda \\
& \leq \frac{1}{\sqrt{8\pi}} \left( \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} d\lambda \right) \left( \int_{\mathbb{R}} e^{-(1-2\mu) x^2} dx \right) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}.
\end{align*}

Evaluating the integrals appearing here yields the result. \qed
5.4.3. **Refined multilinear estimates.** As for the quintic resonant equation, the refined multilinear estimates we need can be determined in an elementary way using the representations (5.6) and (5.8) for $E_4$.

**Lemma 5.11.** There is an absolute constant $C$ such that if $f_1$ and $f_3$ are supported in $B(0,R)^C$ and $f_2$ and $f_4$ are supported in $B(0,r)$, with $R > 4r$, then

\[
|E_4(f_1, f_2, f_3, f_4)| \leq \frac{C}{\sqrt{R}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}
\]

**Proof.** From (5.6) we have,

\[
E_4(f_1, f_2, f_3, f_4) = \frac{1}{2\pi} \int_{\mathbb{R}^4} e^{-\frac{1}{2}(|\lambda v_3|^2 + v_1^2)} f_1(\lambda v_1 + v_3) f_2(v_2 + v_3) f_3(\lambda v_1 + v_2 + v_3) f_4(v_3) dv_2 dv_3 d\lambda.
\]

If the integrand here is non-zero, we necessarily have $|v_3| \leq r$, $|v_2 + v_3| \leq r$, and $|\lambda v_1 + v_3| \geq R$. This gives

\[
|\lambda v_1| \geq |\lambda v_1 + v_3| - |v_3| \geq \frac{R}{2} \quad \text{and} \quad |v_2| \leq |v_2 + v_3| + |v_3| \leq r.
\]

We will use these inequalities to impose constraints on $|\lambda v_2 + v_1|$ and $|\lambda v_2 - v_1|$, which are the inputs to the Gaussians in representation (5.8). If we can ensure that these are large, the fast decay of the Gaussians will imply that $E_4$ is small. By inspection, we see that large values of $|\lambda|$ pose a problem, but such large values can be dealt with separately by using the decay of $1/(1 + \lambda^2)$ in (5.8).

**Regime One:** $|\lambda| \leq \sqrt{R}/4$. Observe that,

\[
|\lambda v_2 + v_1| \geq |v_1| - |\lambda v_2| \geq \frac{R}{2|\lambda|} - 2|\lambda|,
\]

in the last step using (5.31). Because the function $x \mapsto R/(2x) - 2x$ is decreasing for positive $x$, we have,

\[
A := \frac{1}{\sqrt{2\pi}} \int_{|\lambda| \leq \sqrt{R}/4} \frac{1}{1 + \lambda^2} |E_{B(\lambda)}(G, f_1, f_2, G, f_3, f_4)| d\lambda
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{|\lambda| \leq \sqrt{R}/4} \frac{1}{1 + \lambda^2} |E_{B(\lambda)}(G \chi_{|x| \geq \sqrt{R}} f_1, f_2, G \chi_{|x| \geq \sqrt{R}} f_3, f_4)| d\lambda
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \left( \int_{R} \frac{1}{1 + \lambda^2} d\lambda \right) \|G \chi_{|x| \geq \sqrt{R}}\|_{L^2}^2 \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}.
\]

We estimate,

\[
\|G \chi_{|x| \geq \sqrt{R}}\|_{L^2}^2 = 2 \int_{\sqrt{R}} e^{-x^2} dx \leq \frac{2}{\sqrt{R}} \int_{\sqrt{R}} x e^{-x^2} dx = \frac{2}{\sqrt{R}} e^{-R} \leq \frac{2}{\sqrt{R}},
\]

and hence,

\[
A \leq \frac{C}{\sqrt{R}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2},
\]

for some absolute constant $C$.

**Regime Two:** $|\lambda| \geq \sqrt{R}/4$. This is easier: we have,

\[
B := \frac{1}{\sqrt{2\pi}} \int_{|\lambda| \geq \sqrt{R}/4} \frac{1}{1 + \lambda^2} |E_{B(\lambda)}(G, f_1, f_2, G, f_3, f_4)| d\lambda
\]

\[
\leq \frac{1}{\sqrt{\pi}} \left( \int_{\sqrt{R}/4} \frac{1}{1 + \lambda^2} d\lambda \right) \|G\|_{L^2}^2 \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}.
\]
We estimate
\[
\int_{\sqrt{R}/4}^{\infty} \frac{1}{1 + \lambda^2} d\lambda \leq \int_{\sqrt{R}/4}^{\infty} \frac{1}{\lambda^2} d\lambda \leq \frac{4}{\sqrt{R}},
\]
which then gives,
\[
B \leq \frac{C}{\sqrt{R}} ||f_1||_{L^2}||f_2||_{L^2}||f_3||_{L^2}||f_4||_{L^2}.
\]
Then, because \(E_4(f_1, f_2, f_3, f_4) = A + B\), equation (5.29) is established.

**Theorem 5.12.** There is an absolute constant \(C\) such that if \(\mu \in [0, 1/2)\), \(f_k\) is supported in \(B(0, r)\) and \(f_j\) is supported in \(B(0, R)^C\), with \(R > 4r\), then
\[
(5.32) \quad |E_4^\mu(f_1, f_2, f_3, f_4)| \leq \frac{C}{(1 - 2\mu)^{5/8} R^{1/4}} ||f_1||_{L^2}||f_2||_{L^2}||f_3||_{L^2}||f_4||_{L^2}
\]

**Proof.** For fixed \(\mu \in [0, 1/2)\), let \(\alpha = \sqrt{(1/2) - \mu}\). We will again adopt the notation \(f^\alpha(x) = \lambda^{1/2} f(\lambda x)\).

With this notation we have\
\[
e^{-\frac{1}{2} - \mu)\lambda^2} = e^{-\alpha x^2} = G(\alpha x) = \alpha^{-1/2} G^\alpha(x).
\]

Using the scaling property of \(E_B(\lambda)\), we have,
\[
E_4^\mu(f_1, f_2, f_3, f_4) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} E_B(\lambda) \left( \alpha^{-1/2} G^\alpha, f_1, f_2, \alpha^{-1/2} G^\alpha, f_3, f_4 \right)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} E_B(\lambda) \left( G, f^1_1, f^1_2, f^1_3, f^1_4 \right)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} E_B(\lambda) \left( f^1_1, f^1_2, f^1_3, f^1_4 \right).
\]

Now assume that \(f_{k_1}\) is supported in \(B(0, R)^C\) and \(f_{k_2}\) is supported in \(B(0, r)\). We then have that \(f^1_{k_1}\) is supported in \(B(0, \alpha R)^C\) and \(f^1_{k_2}\) is supported in \(B(0, \alpha r)\). Then, using representation (5.2), we have
\[
|E_4^\mu(f_1, f_2, f_3, f_4)|
\]
\[
= \left| \frac{1}{\alpha^2} \mathcal{E}_4(f^1_{k_1}, f^1_{k_2}, f^1_{k_3}, f^1_{k_4}) \right|
\]
\[
= \frac{1}{\alpha} \int_0^{\pi/2} \int_{\mathbb{R}} \left( e^{itH} f^1_{k_1} f^1_{k_2} f^1_{k_3} f^1_{k_4} \right) dt dx
\]
\[
\leq \frac{1}{\alpha} \left( \frac{2}{\pi} \int_0^{\pi/2} \int_{\mathbb{R}} \left( |e^{itH} f^1_{k_1} f^1_{k_2} f^1_{k_3} f^1_{k_4}|^2 \right)^{1/2} \left( \frac{2}{\pi} \int_0^{\pi/2} \int_{\mathbb{R}} \left( |e^{itH} f^1_{k_1} f^1_{k_2} f^1_{k_3} f^1_{k_4}|^2 \right)^{1/2} \right)^{1/2}
\]
\[
\frac{1}{\alpha^2} \mathcal{E}_4(f^1_{k_1}, f^1_{k_2}, f^1_{k_3}, f^1_{k_4})^{1/2} \mathcal{E}_4(f^1_{k_1}, f^1_{k_2}, f^1_{k_3}, f^1_{k_4})^{1/2}
\]

Using (5.29), we get
\[
\mathcal{E}_4(f^1_{k_1}, f^1_{k_2}, f^1_{k_3}, f^1_{k_4})^{1/2} \leq \frac{C}{(\alpha R)^{1/2}} ||f^1_{k_1}||_{L^2} ||f^1_{k_2}||_{L^2} \Rightarrow \frac{C}{(\alpha R)^{1/2}} ||f^1_{k_1}||_{L^2} ||f^1_{k_2}||_{L^2},
\]
while for the other \(\mathcal{E}_4\) term we can use the usual \(L^2\) boundedness. This gives
\[
|E_4^\mu(f_1, f_2, f_3, f_4)| \leq \frac{C}{\sqrt{R}} ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^2} ||f_4||_{L^2}.
\]

Substituting back in \(\alpha = \sqrt{(1/2) - \mu}\) gives the result. \(\square\)
5.4.4. Stationary waves are analytic.

**Theorem 5.13.** Suppose that $\phi \in L^2$ is a stationary wave solution of $iu_t = T_4(u, u, u)$; that is, $\phi$ satisfies
\begin{equation}
(5.33) \quad \omega \phi(x) = T_4(\phi, \phi, \phi)(x),
\end{equation}
for some $\omega$. Then there exists $\alpha > 0$ and $\beta > 0$ such that $\phi e^{\alpha x^2} \in L^\infty$ and $\hat{\phi} e^{\beta x^2} \in L^\infty$. As a result, $\phi$ can be extended to an entire function on the complex plane.

Using the proof of Corollary 4.16, this theorem is an immediate consequence of the following proposition.

**Proposition 5.14.** Suppose that $\phi \in L^2$ satisfies
\begin{equation}
(5.34) \quad \omega |\phi(x)| \leq T_4(|\phi|, |\phi|, |\phi|)(x),
\end{equation}
for some $\omega > 0$. Then there exists $\alpha > 0$ such that $x \mapsto \phi(x)e^{\alpha x^2} \in L^2$.

**Proof of proposition.** For the proof, we will find $\mu$ so that we have the bound $\|\phi G_{\mu, \epsilon}\|_{L^2} \lesssim 1$ independently of $\epsilon$. Taking the limit $\epsilon \to 0$ will then yield the result. The structure of proof here is extremely similar to that of Theorem 4.15. For brevity, we will only describe the start of the proof here, which is the only part that is essentially different to the proof of Theorem 4.15.

First, we fix throughout $\mu \leq 1/4$. Using formulas (5.27), (5.28) and (5.32), there are constants $C$ independent of $\mu$, such that,
\begin{align}
(5.35) & \quad \mathcal{E}_4(f_1, f_2, f_3, f_4 G_{\mu, \epsilon}) \leq \mathcal{E}_4^\mu(f_1 G_{\mu, \epsilon}, f_2 G_{\mu, \epsilon}, f_3 G_{\mu, \epsilon}, f_4) \\
(5.36) & \quad |\mathcal{E}_4^\mu(f_1, f_2, f_3, f_4)| \leq C \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2} \\
(5.37) & \quad |\mathcal{E}_4^\mu(f_1, f_2, f_3, f_4)| \leq C \frac{1}{R^{1/4}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2},
\end{align}
where in the last inequality, $f_i$ is supported in $B(0, r)$ and $f_j$ is supported in $B(0, R)^C$ for $R > 4r$.

Now consider a function $\phi$ satisfying (5.34). We may assume $\phi$ is non-negative. For any $M > 0$ define,
\begin{align*}
\phi_<(x) &= \phi(x) \chi_{|x| \leq M}(x), \\
\phi_>(x) &= \phi(x) \chi_{|x| > M}(x), \\
\phi_{\sim}(x) &= \phi(x) \chi_{M < |x| < M^2}(x).
\end{align*}

We have the decomposition $\phi = \phi_+ + \phi_- + \phi_\sim$, and the supports are all disjoint, which gives
\begin{align*}
\|\phi G_{\mu, \epsilon}\|_{L^2}^2 &= \|\phi_+ G_{\mu, \epsilon}\|_{L^2}^2 + \|\phi_- G_{\mu, \epsilon}\|_{L^2}^2 + \|\phi_\sim G_{\mu, \epsilon}\|_{L^2}^2.
\end{align*}

The first two terms are trivial to to bound uniformly in $M$. If $|x| \leq M^2$, we have,
\begin{align*}
G_{\mu, \epsilon}(x) &\leq e^{4\mu x^2} \leq e^{M^4}
\end{align*}
so setting $\mu = M^{-4}$ gives $\|\phi_+ G_{\mu, \epsilon}\|_{L^2} \leq \|\phi_\sim e^1\|_{L^2} \leq e^1 \|\phi\|_{L^2} \lesssim 1$, with the same bound for $\phi_-$. In order to prove the theorem, it remains to bound $\|\phi_\sim e^{G_{\mu, \epsilon} \cdot \cdot \cdot}\|_{L^2}$.

Starting with the equation (5.34) of the theorem, we multiply both sides by $\phi_\sim(x) G_{\mu, \epsilon}(x)^2$ which gives
\begin{align*}
\omega \phi_\sim(x)^2 G_{\mu, \epsilon}(x)^2 &\leq T_4(\phi, \phi, \phi_\sim(x) G_{\mu, \epsilon}(x)^2)
\end{align*}
Now integrating over $\mathbb{R}$ and using (5.35) gives,
\begin{align*}
\omega \|\phi_\sim G_{\mu, \epsilon}\|_{L^2}^2 &\leq \mathcal{E}_4(\phi, \phi, \phi_\sim G_{\mu, \epsilon}) \leq \mathcal{E}_4^\mu(\phi G_{\mu, \epsilon}, \phi G_{\mu, \epsilon}, \phi G_{\mu, \epsilon}^2).
\end{align*}

For convenience, let $\psi = \phi G_{\mu, \epsilon}$. The bound then reads
\begin{align*}
\omega \|\psi\|_{L^2}^2 &\lesssim \mathcal{E}_4^\mu(\psi, \psi, \psi, \psi_\sim).
\end{align*}

Now write each $\psi = \psi_\sim + \psi_+ + \psi_\sim$ and expand the multilinear functional. We will get many terms, which we bound in one of two ways.

- If there are three or more $\psi_\sim$ terms, bound by $\|\psi\|_{L^2}^k$ where $k$ is the number of $\psi_\sim$ terms appearing, using (5.36). In this case the other terms are $\psi_\sim$ or $\psi_\sim$, which are uniformly bounded.
If there are one or two $\psi_>$ terms, then there is either a $\psi_<$ term or a $\psi_.$ term. In the former case we can use the refined multilinear estimate (5.37), with $R = M^2$, and bound by $M^{-1/2}\|\psi_>\|^k$ (where $k = 1$ or $k = 2$). In the latter case we can bound by $\|\psi_\sim\|_{L^2}\|\psi_\sim\|_{L^2}^k \lesssim \|\phi_\sim\|_{L^2}\|\psi_\sim\|_{L^2}^k$ using (5.35).

In total, we get,

$$\omega \|\psi_\sim\|_{L^2} \leq E_A^B(\psi, \ldots, \psi) \leq C \left(\|\psi_\sim\|_{L^2}^2 + \|\psi_\sim\|_{L^2}^2 + \left(M^{-1/2} + \|\phi_\sim\|_{L^2}\right)^{\|\psi_\sim\|_{L^2}^2 + \|\psi_\sim\|_{L^2}}\right),$$

for a constant $C$ independent of $\mu$. This formula has the same structure as equation (4.42) in the proof of Theorem 4.15. Replicating the same argument there, we find that if we choose $M$ sufficiently large there is a constant independent of $\epsilon$ such that $\|\psi_\sim\|_{L^2} \leq C$. Letting $\epsilon \to 0$ then gives the result. □

Appendix A. Proof of Theorem 3.12

Theorem 3.12 has two distinct parts, which we state here independently as Lemma A.1 and Theorem A.2.

**Lemma A.1** (Decomposition Lemma). Let $A$ be an isometry. Denote $f_k(x) = f_k(-x)$. There exists integers $m$ and $l$, with $0 \leq m \leq l \leq n$, and two permutations $\sigma_1$ and $\sigma_2$ of the integers $\{1, \ldots, n\}$ such that,

$$E_A(f_1, \ldots, f_{2n}) = \left(\prod_{k=1}^{m} \langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)} \rangle\right) \left(\prod_{k=m+1}^{l} \langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)} \rangle\right) \times E_B(f_{\sigma_1(l+1)} \ldots f_{\sigma_1(n)}, f_{n+\sigma_2(l+1)} \ldots f_{n+\sigma_2(n)}),$$

where the matrix $B : \mathbb{R}^{n-l} \to \mathbb{R}^{n-l}$ has no permutation part; that is, for all $i$ and $j$, $Be_i \neq \pm e_j$.

**Proof.** Call a pair of integers $(i, j)$ good if $Ae_i = e_j$ and bad if $Ae_i = -e_j$. Because $A$ is injective, for a given $j$ there is at most one $i$ such that $(i, j)$ is good or bad. Let $m$ be the number of good pairs, and $l - m$ the number of bad pairs. Order the good pairs in any way, and for $k = 1, \ldots, m$, let $\sigma_1(k) = i$ and $\sigma_2(k) = j$ where $(i, j)$ is the $k$th good pair in the ordering. In a similar fashion, order the bad pairs in any way, and for $k = l + 1, \ldots, m$, let $\sigma_1(k) = i$ and $\sigma_2(k) = j$ where $(i, j)$ is the $(k - m)$th bad pair in the ordering.

Now consider $i$ such that $i$ is not the first component in a good or bad pair. There are $n - l$ such $i$. Order them in any way, and for $k = l + 1, \ldots, n$ set $\sigma_1(k) = i$ where $i$ is the $(k - l)$th number in the ordering. Then consider $j$ such that $j$ is not the second component in a good or bad pair, and for $k = l + 1, \ldots, n$ define $\sigma_2(k) = j$ in a similar fashion.

It is clear that,

(A.1) \quad $A : \text{span}(e_{\sigma_1(1)}, \ldots, e_{\sigma_1(m)}) \cong \mathbb{R}^m \to \text{span}(e_{\sigma_2(1)}, \ldots, e_{\sigma_2(m)}) \cong \mathbb{R}^m$

(A.2) \quad $A : \text{span}(e_{\sigma_1(m+1)}, \ldots, e_{\sigma_1(l)}) \cong \mathbb{R}^{l-m} \to \text{span}(e_{\sigma_2(m+1)}, \ldots, e_{\sigma_2(l)}) \cong \mathbb{R}^{l-m}$,

and that $A = I$ and $A = -I$ on these subspaces respectively, with the implied identifications of bases. Because $A$ is an isometry, we then necessarily have,

(A.3) \quad $A : \text{span}(e_{\sigma_1(m+1)}, \ldots, e_{\sigma_1(n)}) \cong \mathbb{R}^{n-m} \to \text{span}(e_{\sigma_2(m+1)}, \ldots, e_{\sigma_2(n)}) \cong \mathbb{R}^{n-m}$.

Let $B$ be the restriction of $A$ as a map between the subspaces in (A.3). The condition that $Be_i \neq \pm e_j$ holds because otherwise $(i, j)$ would be a good or bad pair and $e_i$ and $e_j$ would not be in the subspaces given in (A.3).

The representation of $E_A$ arises because $A$ is the identity in (A.1) and the negative of the identity in (A.2). □

In light of the representation of $E_A$, it is clear that we have the equality $|E_A(f_1, \ldots, f_{2n})| = \prod_{k=1}^{2n} \|f_k\|_{L^2}$ if and only if the following three conditions hold.

1. For all $k = 1, \ldots, m$, $\|f_{\sigma_1(k)}\|_{L^2} \leq \|f_{\sigma_1(k)}\|_{L^2}\|f_{n+\sigma_2(k)}\|_{L^2}$, which means $f_{\sigma_1(k)} = C_k f_{n+\sigma_2(k)}$ for some constant $C_k$ by the usual Cauchy–Schwarz equality condition.
(2) For all \( k = m+1, \ldots, l \), \( |\langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)} \rangle| = \|f_{\sigma_1(k)}\|_{L^2} \|f_{n+\sigma_2(k)}\|_{L^2} \), which means \( f_{\sigma_1(k)} = C_k f_{n+\sigma_2(k)} \) for some constant \( C_k \), again by the usual Cauchy–Schwarz equality condition.

(3) \( |E_B(f_{\sigma_1(1)}, \ldots, f_{\sigma_1(n)}, f_{n+\sigma_2(1)}, \ldots, f_{n+\sigma_2(n)})| = \prod_{k=l+1}^n \|f_{\sigma_1(k)}\|_{L^2} \|f_{n+\sigma_2(k)}\|_{L^2} \)

To finish the proof of Theorem 3.12, we examine the equality case in item 3. This equality is, of course, identical looking to the original equality condition (3.19). The difference is that \( B \) has the structural condition \( B_{ei} \neq \pm e_j \).

**Theorem A.2.** Suppose that \( A \) is an isometry, and for all \( i \) and \( j \), \( Ace_i \neq \pm e_j \). Then,

\[
|E_A(f_1, \ldots, f_{2n})| = \prod_{k=1}^{2n} \|f_k\|_{L^2},
\]

only if each of the functions is a Gaussian. Equality holds if each of the functions is the same Gaussian of the form \( e^{-\alpha x^2} \) for some \( \alpha > 0 \).

The ‘if’ part of the Theorem was proved before, in Theorem 3.11. The ‘only if’ part follows from a close analysis of the Cauchy–Schwarz equality condition,

**Lemma A.3 (Step One).** Suppose that functions \( f_k \) satisfy (A.4), are all strictly positive and smooth. Then they are all Gaussians.

**Proof.** Let \( g_k(x) = \log f_k(x) \). Because \( f_k \) is strictly positive and smooth, \( g_k \) is well-defined and smooth. To show \( f_k \) is a Gaussian we will show that \( g_k \) is polynomial of degree at most two.

Because \( A \) is an isometry, it satisfies \( A^{-1} = A^T \). Therefore, for all \( m \), we have the matrix expansions,

\[
(A.5)
Ae_m = \sum_{k=1}^n a_{km} e_k \quad \text{and} \quad A^{-1}e_m = \sum_{k=1}^n a_{mk} e_k,
\]

for numbers \( \{a_{km}\}_{k,m=1}^n \). By the assumptions in the theorem, we have \( |a_{km}| < 1 \) for all \( k \) and all \( m \). Set \( \epsilon = \max_{m,k} |a_{km}| < 1 \). Because \( A \) is an isometry, we have,

\[
(A.6)
1 = |Ae_m|^2 = \sum_{k=1}^n a_{km}^2 \quad \text{and} \quad 1 = |A^{-1}e_m|^2 = \sum_{k=1}^n a_{mk}^2,
\]

for all \( m \).

Taking the logarithm of both sides of the Cauchy–Schwarz equality condition (A.4) (and switching the left and right sides) gives the condition on the functions \( g_k \),

\[
(A.7)
\sum_{k=1}^n g_{n+k}(x_k) = \sum_{k=1}^n g_k((Ax)_k).
\]
Fix $m \in \{1, \ldots, n\}$ and set $x = t e_m$. For all $k \in \{1, \ldots, n\}$, we have $(Ax)_k = t(Ae_m)_k = ta_{mk}$ and $x_k = t \delta_{km}$. Therefore,

$$g_{n+m}(t) + \sum_{k=1 \atop k \neq m}^{n} g_{n+k}(0) = \sum_{k=1}^{n} g_k(a_{km}t).$$

Differentiating this equation twice with respect to $t$ yields,

$$g''_{n+m}(t) = \sum_{k=1}^{n} a_{km}^2 g_k''(a_{km}t).$$

We can evaluate this equation at $t = 0$ to get $g''_{n+m}(0) = \sum_{k=1}^{n} a_{km}^2 g_k''(0)$. Subtracting this from (A.8) then gives,

$$g''_{n+m}(t) - g''_{n+m}(0) = \sum_{k=1}^{n} a_{km}^2 [g_k''(a_{km}t) - g_k''(0)].$$

For all indices $m$ and $k$ we have $a_{km}t \in [-\epsilon|t|, \epsilon|t|]$ and so the bound,

$$|g''_{n+m}(t) - g''_{n+m}(0)| \leq \max_{s \in [-\epsilon|t|, \epsilon|t|]} \max_{k=1, \ldots, n} |g_k''(s) - g_k''(0)|,$$

holds for all $m \in \{1, \ldots, n\}$.

The Cauchy–Schwarz equality condition is similar when the roles of $f_1, \ldots, f_n$ and $f_{n+1}, \ldots, f_{2n}$ are switched. This may be seen by replacing $x$ by $Ax$ and finding,

$$\prod_{k=1}^{n} f_{n+k}((A^{-1}x)_k) = \prod_{k=1}^{n} f_k(x_k),$$

By equations (A.5) and (A.6), the same estimates for $A$ hold for $A^{-1}$. Therefore, performing the same argument as before, we find,

$$|g''_m(t) - g''_m(0)| \leq \max_{s \in [-\epsilon|t|, \epsilon|t|]} \max_{k=1, \ldots, n} |g_k''(s) - g_k''(0)|$$

for all $m \in \{1, \ldots, n\}$.

The two families of inequalities (A.10) and (A.12) may be combined into one inequality,

$$\max_{k=1, \ldots, 2n} |g_k''(t) - g_k''(0)| \leq \max_{s \in [-\epsilon|t|, \epsilon|t|]} \max_{k=1, \ldots, 2n} |g_k''(s) - g_k''(0)|$$

This holds for all $t$. Applying it recursively $N$ times yields

$$\max_{k=1, \ldots, 2n} |g_k''(t) - g_k''(0)| \leq \max_{s \in [-\epsilon^N|t|, \epsilon^N|t|]} \max_{k=1, \ldots, 2n} \lim_{N \to \infty} \max_{s \in [-\epsilon^N|t|, \epsilon^N|t|]} \max_{k=1, \ldots, 2n} |g_k''(s) - g_k''(0)|$$

But now $\epsilon < 1$, and so we have, because $g$ is smooth,

$$\max_{k=1, \ldots, 2n} |g_k''(t) - g_k''(0)| \leq \lim_{N \to \infty} \max_{s \in [-\epsilon^N|t|, \epsilon^N|t|]} \max_{k=1, \ldots, 2n} |g_k''(s) - g_k''(0)| = 0.$$

Hence $g_k''(x) = g_k''(0)$ is a constant, $g_k(x)$ is a polynomial of degree at most two, and $f_k(x) = e^{g_k(x)}$ is a Gaussian. \hfill \Box

The proof of the previous lemma required smoothness and positivity assumptions on the $f_k$ functions. We now use a heat flow argument to upgrade the result to non-smooth and non-negative $f_k$.

**Lemma A.4** (Step two, part one). The Cauchy–Schwarz equality condition (A.A) is conserved by the heat flow. More precisely, suppose that there is a $\gamma > 0$ such that all of the functions in (A.A) satisfy $f_k e^{-\gamma x^2} \in L^1$. Then there is a time $T$ such that the heat flow $e^{-t \Delta} f_k$ is defined for $t \in (0, T)$ and all $k$. For fixed $t \in (0, T)$ the functions $h_k(x) = (e^{-t \Delta} f_k)(x)$ satisfy (A.A).
Proof. Under the assumptions on $f_k$ we can write the solution of the heat equation $e^{-t\Delta}f_k$ using the fundamental solution as,

$$(e^{-t\Delta}f_k)(x_k) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} f(x_k - y_k) dy.$$ 

The formula is well defined for $0 < t < T$, with $T < 1/\gamma$, because of the integrability assumptions on $f_k$. Using this formula $n$ times, we have,

$$\prod_{k=1}^{n} (e^{-t\Delta}f_k)((Ax)_k) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} \prod_{k=1}^{n} f_k((Ax)_k - y_k) dy$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} \prod_{k=1}^{n} f_k((A(x - A^{-1}y))_k) dy$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} \prod_{k=1}^{n} f_{n+k}((x - A^{-1}y)_k) dy,$$

where in the last line we have used the Cauchy–Schwarz equality condition (A.4). We now perform the change of variables $z = A^{-1}y$. Because $A$ is an isometry, the determinant of this change of variables is 1, and we also have $|y| = |Az| = |z|$ for every $y \in \mathbb{R}^n$. This gives,

$$\prod_{k=1}^{n} (e^{-t\Delta}f_k)((Ax)_k) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2/4t} \prod_{k=1}^{n} f_{n+k}(x_k - z_k) dy = \prod_{k=1}^{n} (e^{-t\Delta}f_{n+k})(x_k).$$

For fixed $t$ the maps $h_k(x) = e^{-t\Delta}f(x)$ thus satisfy (A.4).

Corollary A.5 (Part two, step two). Suppose that non-negative functions $f_k \in L^2$ satisfy (A.4). Then all of the functions are Gaussians.

Proof. If $f_k \in L^2$, then $f_k e^{-\gamma^2 z^2} \in L^1$ for all $\gamma > 0$. By the previous lemma, the heat flow $e^{-t\Delta}f_k$ exists for all $k$ and $t \in (0, \infty)$, and the functions $h_k(x) = e^{-t\Delta}f_k(x)$ satisfy (A.4). The functions $h_k$ are smooth, and because the initial data is non-negative, the functions $h_k$ are also positive. By the first lemma, each of the $h_k$ functions are Gaussians.

Fix $k$. By substituting the general form of a time dependent Gaussian into the heat equation we discover that if $e^{t\Delta}f_k$ is a Gaussian for all $t \in (0, T)$ then necessarily, for $t > 0$,

$$(e^{-t\Delta}f_k)(x) = \frac{d}{\sqrt{t + a}} e^{-b(x-c)^2/4(t+a)},$$

for some $a \geq 0$ and $b, d > 0$ and $c \in \mathbb{R}$. We calculate,

$$\|e^{-t\Delta}f_k\|_{L^2}^2 = \frac{d^2}{t + a} \int_{\mathbb{R}^n} e^{-b(x-c)^2/2(t+a)} dx = \frac{d^2}{\sqrt{t + a}} \sqrt{\frac{2\pi}{b}}.$$ 

Then, because $\|e^{-t\Delta}f_k\|_{L^2} \leq \|f_k\|_{L^2}$, we must have $a > 0$. This gives

$$f_k(x) = \lim_{t \to 0} (e^{-t\Delta}f_k)(x) = \frac{1}{\sqrt{a}} e^{(bx^2+cx)/4a},$$

so $f_k$ is a Gaussian. 

Having proved the result for non-negative $f_k$ we lastly prove it for general complex valued $f_k$.

Lemma A.6 (Step three). Suppose that functions $f_k \in L^2$ satisfy (A.4). Then all of the functions are Gaussians.

Proof. We assume that the functions $f_k$ are smooth; the result for general functions follows from invoking the heat flow argument as in Step Two.

Taking absolute values in the equality condition (A.4), we see that the functions $|f_k|$ satisfy the condition as soon as the functions $f_k$ do. By the previous Lemmas, $|f_k|$ must be a Gaussian $G_k(x)$ for all $k$, and hence

$$f_k(x) = e^{ig_k(x)} G_k(x),$$

(A.14)
for some real valued function $g_k$. Note that from this we necessarily have $f_k(x) \neq 0$. Because $f_k$ and $G_k$ are smooth, and $e^{ig_k(x)} = f_k(x)/G_k(x)$ we can choose $g_k$ to be smooth.

Plugging the expressions (A.14) for $f_k$ into (A.4) we find

$$\prod_{k=1}^{n} e^{ig_k((Ax)_k)} G_k((Ax)_k) \prod_{k=1}^{n} e^{ig_{n+k}(x_k)} G_{n+k}(x_k).$$

The $G_k$ terms cancel because we know the functions $G_k = |f_k|$ also satisfy (A.4). We are left with $e^{\sum_{k=1}^{n} g_k((Ax)_k)} = e^{\sum_{k=1}^{n} g_k(x_k)}$, and hence

$$\sum_{k=1}^{n} g_k((Ax)_k) = \sum_{k=1}^{n} g_{n+k}(x_k) + 2\pi n(x).$$

where $n : \mathbb{R}^n \to \mathbb{Z}$. Because $g$ is smooth, $n$ is smooth and hence a constant. Plugging in $x = 0$ gives $n(x) = 0$, and then the equation for the functions $g_k$ is thus,

$$\sum_{k=1}^{n} g_k((Ax)_k) = \sum_{k=1}^{n} g_{n+k}(x_k).$$

This is precisely equation (A.7). As before, $g_k$ must be a polynomial of degree at most two, and $f_k$ is a complex Gaussian. 

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