A greedy, partially optimal proof of the Heine-Borel Theorem

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Abstract

The Heine-Borel theorem states that any open cover of the closed interval \([a, b]\) contains a finite subcover. We give a new, constructive proof inspired by the concept of a greedy algorithm. In the case when the initial cover consists only of intervals, our proof is optimal in that it constructs a finite subcover with the fewest possible number of elements. Our work leaves open a number of questions which are discussed in the final section.

1 Introduction

The Heine-Borel theorem is a fundamental theorem in real analysis which, in its most general form, asserts that every closed and bounded subset of \(\mathbb{R}^n\) is compact. In this note we discuss the special case of the theorem regarding the compactness of the interval \([a, b]\). Historically, this was the first case of the theorem to be considered and proved (see [3] for a wonderfully comprehensive account of the history of the compactness concept). Often it is the first case of the theorem that undergraduate students encounter – indeed in many textbooks the compactness of \([a, b]\) is presented as the Heine-Borel theorem [2]. Before describing our small contribution to the story, we recall the formal statement of the result.

Definition 1. An open cover \(\mathcal{O}\) of \([a, b]\) is a collection of open sets \(\mathcal{O} = \{O_a\}_{a \in A}\) such that \([a, b] \subseteq \bigcup_{a \in A} O_a\).

A subcover of \(\mathcal{O}\) (with respect to \([a, b]\)) is a subset of \(\mathcal{O}\) that also covers \([a, b]\). A finite subcover is a subcover that contains a finite number of elements.

Theorem 1 (Heine-Borel). Every open cover of \([a, b]\) contains a finite subcover.

The standard proofs of Theorem 1 are short, slick, but completely non-constructive. (The textbooks [1] and [4] contain different proofs; other books consulted contained one of those two.) The present article originated with an attempt to find a constructive proof. Along the way we discovered that the argument we developed, which appears to be new, has the following curious optimality feature: if the initial open cover consists of open intervals, then the subcover constructed has the fewest number of elements possible. In other words, if the subcover constructed in the proof
below contains $N$ elements, then no subcover exists that contains $N - 1$ or fewer elements.

While our proof implies that every closed and bounded subset of $\mathbb{R}$ is compact, we show in the final section that the optimality part of our proof only applies in general when the closed and bounded subset is an interval and the open cover consists of open intervals. It would be interesting to extend the present result by finding a constructive process to determine the optimal subcover in the general case, and in higher dimensions.

Before moving to the technical presentation of our proof of Theorem 1, we briefly describe its inspiration and features. The proof was motivated by the concept of a greedy algorithm in computer science. The proof assumes (without loss of generality) that the open cover consists only of intervals. Starting at the left endpoint $x_1 = a$, the idea is to consider all intervals in the open cover that contain $x_1 = a$, and greedily choose the one that has the largest right endpoint $x_2$. Then, among all intervals containing $x_2$, choose the interval with the largest right endpoint $x_3$. Continuing this process recursively, one hopes that after some finite number of steps it concludes and yields a finite subcover.

The flaw with this argument is that the open cover $\mathcal{O}$ is in general infinite, and hence there is no such thing in general as the interval containing $a$ with the largest right endpoint. The solution is to consider the supremum of all right endpoints of those intervals that contain $a$, which at least gives a definition of $x_2$. Repeating this supremum process recursively, we construct a sequence of points in $[a, b]$ that is provably finite. After the fact of finding these points, we are able to use some of their properties to construct a finite subcover of $\mathcal{O}$. We conclude the proof by showing that the subcover so constructed is the smallest possible; like the analysis of many greedy algorithms, this part is actually quite straightforward.

2 The algorithms and the proof

We are given an interval $[a, b]$ and an open cover $\mathcal{O}$ of $[a, b]$ and our task is to construct a finite subcover of $\mathcal{O}$.

First, we remark that we may assume without loss of generality that the open cover $\mathcal{O}$ consists entirely of intervals. To see this, recall that any open subset of $\mathbb{R}$ can be written as the countable union of intervals. Given an open cover $\mathcal{O}$ consisting of general open sets, an open cover $\mathcal{O}'$ consisting of intervals may be found by decomposing each open set into the associated countable set of intervals. Having determined a finite subcover of $\mathcal{O}'$, a finite subcover of $\mathcal{O}$ is determined by choosing those elements of $\mathcal{O}$ that have an associated interval in the finite subcover of $\mathcal{O}'$.

With this assumption in place we now move to the proof. In a nod to our inspiration from computer science, we present the constructive part of our proof in the form of two algorithms. The first algorithm selects the sequence of special points, and the second selects a finite subcover using those points.

Algorithm 1 (Selection of points). Given an interval $[a, b]$ and an open cover $\mathcal{O}$ of $[a, b]$, construct a sequence of points $\{x_n\}_{n=1,2,3,...}$ as follows.
• **Step one.** Set $x_1 = a \in [a, b]$.

• **Step two.** Given $x_k \in [a, b]$, define $M_k$ by

$$M_k = \sup \{x_k + \beta : (x_k - \alpha, x_k + \beta) \in \mathcal{O}, \alpha, \beta > 0\}.$$  \hspace{1cm} (1)

If $M_k \leq b$, set $x_{k+1} = M_k \in [a, b]$ and perform step two for $x_{k+1}$. Otherwise, terminate and return $\{x_n\}_{n=1}^k$.

We make two remarks. First, because $\mathcal{O}$ is an open cover of $[a, b]$ and $x_n \in [a, b]$ in step two, the set of intervals in $\mathcal{O}$ containing $x_n$ is non-empty and so the supremum in (1) is well-defined. Second, because there is at least one interval $(x_n - \alpha, x_n + \beta)$ containing $x_n$, we have $M_n \geq x_n + \beta > x_n$. From this it follows that the sequence of points constructed in Algorithm 1 is strictly increasing; that is, $x_n < x_{n+1}$.

The monotonicity of the sequence is used to prove the first Lemma, which at first glance may not appear to be very useful but is in fact the key in everything that follows.

**Lemma 1.** Suppose $x_n$ and $x_m$ are distinct. Then there is no element of $\mathcal{O}$ that contains both $x_n$ and $x_m$.

**Proof.** We may assume without loss of generality that $n < m$. Because the sequence is strictly increasing, this implies that $x_n < x_m$.

Suppose, for a contradiction, that there is an interval $I \in \mathcal{O}$ that contains both $x_n$ and $x_m$. Because $I$ contains $x_n$, we can write it as $I = (x_n - \alpha, x_n + \beta)$ for some positive numbers $\alpha$ and $\beta$. Because $I$ also contains $x_m$, we have $x_m < x_n + \beta$. But now by the definition of $x_{n+1} = M_n$ in (1) we have $x_m < x_n + \beta \leq x_{n+1}$. This gives $x_n < x_m < x_{n+1}$, which contradicts the monotonicity of the sequence. \hfill \Box

**Lemma 2.** Algorithm 1 terminates after a finite number of iterations and hence produces a finite sequence of points.

**Proof.** Suppose, for a contradiction, that the algorithm never terminates and instead constructs an infinite sequence $\{x_n\}_{n=1}^{\infty}$. This sequence is monotonically increasing, and bounded because $x_n \leq b$. The sequence thus has a limit $L$. From $x_n \leq b$ we have $L \leq b$, and so $L \in [a, b]$. Because $\mathcal{O}$ is an open cover of $[a, b]$, there is an interval $(L - \alpha, L + \beta) \in \mathcal{O}$. By the definition of $L = \lim_{n \to \infty} x_n$, we have $x_n \in (L - \alpha, L + \beta)$ for all sufficiently large $n$. In particular, there are multiple elements of the sequence $\{x_n\}_{n=1}^{\infty}$ in the set $(L - \alpha, L + \beta) \in \mathcal{O}$, contradicting Lemma 1. \hfill \Box

We will denote the last element in the finite sequence produced by Algorithm 1 by $x_N$. The sequence produced is thus $\{x_n\}_{n=1}^{N}$. Some of the significance of the sequence may be seen in the following Lemma, which is the first optimality type result.

**Lemma 3.** If $n \leq N$, then any subset $\mathcal{O}'$ of $\mathcal{O}$ that is a cover of $[a, x_n]$ has at least $n$ elements.

**Proof.** The interval $[a, x_n]$ contains the $n$ points $x_1, \ldots, x_n$. By Lemma 1, each of these $n$ points can be associated to a unique element of $\mathcal{O}' \subseteq \mathcal{O}$. The subcover therefore has at least $n$ elements. \hfill \Box
Algorithm 2 (Selection of Intervals). Given an interval \([a, b]\), an open cover \(\mathcal{O}\) of \([a, b]\), and the sequence of points \(\{x_n\}_{n=1}^N\) produced by Algorithm 1, construct a sequence of intervals \(\{O_n\}_{n=1}^N\) as follows.

- **Step one.** Let \(O_N\) be any open interval in \(\mathcal{O}\) containing both \(x_N\) and \(b\). Note that \(x_N \in [a, b]\) and \(M_N > b\) in Algorithm 1, so such an interval necessarily exists.

- **Step two.** Given an open set \(O_{k+1} \in \mathcal{O}\) containing \(x_{k+1}\), construct an open set containing \(x_k\) as follows. The set \(O_{k+1}\) containing \(x_{k+1}\) is of form \((x_{k+1} - \delta, x_{k+1} + \epsilon)\) for positive \(\delta\) and \(\epsilon\). By the definition of the supremum \(x_{k+1} = M_k\) in (1), there is an open interval \((x_{k+1} - \alpha, x_{k+1} + \beta)\) with
  \[
  x_{k+1} - \delta < x_k + \beta. \tag{2}
  \]
  Choose this interval as \(O_k\).

  If \(k > 1\), repeat step two with \(O_k\); otherwise terminate and return \(\{O_n\}_{n=1}^N\).

**Theorem 2.** The set \(\{O_n\}_{n=1}^N\) is a subcover of \(\mathcal{O}\) with \(N\) elements. If \(\mathcal{O}'\) is another subcover of \(\mathcal{O}\), then it has at least \(N\) elements.

**Proof.** We first show that \(\{O_n\}_{n=1}^N\) is a cover of \([a, b]\). Take any \(x \in [a, b]\); we must show that \(x\) is contained in \(O_n\) for some \(n\). If \(x \geq x_N\), then \(x \in [x_N, b]\) and hence by the construction in step one of Algorithm 2, \(x \in O_N\). If \(x < x_N\) then \(x \in [x_n, x_{n+1}]\) for some \(n\). By construction,

\[
O_n = (x_n - \alpha, x_n + \beta) \quad \text{and} \quad O_{n+1} = (x_{n+1} - \delta, x_{n+1} + \epsilon),
\]

with \(x_{n+1} - \delta < x_n + \beta\) from (2). The inequality implies that either \(x \in O_n\) or \(x \in O_{n+1}\).

For the optimality claim, we observe that \([a, x_N] \subseteq [a, b]\). Any subset of \(\mathcal{O}\) that covers \([a, b]\) necessarily covers \([a, x_N]\), and hence by Lemma 2 has at least \(N\) elements.

With our proof of Theorem 2 we have constructively proved Theorem 1.

### 3 Counterexamples to optimality and open questions

Having presented our main result, we will now discuss some of the ways in which the optimality part of it can fail. The construction above produces the smallest possible subcover if we make the assumption that the original open cover \(\mathcal{O}\) consists only of open intervals, We will first consider the case when this assumption does not hold, and present a counterexample to optimality. We will then discuss the generalization of Theorem 1 to the statement that every closed and bounded subset of \(\mathbb{R}\) is compact. Our
proof above extends in a constructive way to a proof of this generalization; however, we are again able to give a counterexample to optimality in this case.

The fact that our constructive proof of Theorem 1 is insufficient to handle the general case in an optimal way naturally suggests the question: given a closed and bounded set \( C \subseteq \mathbb{R} \) and an open cover \( \mathcal{O} \) of \( C \), can one describe a process by which the smallest subcover of \( \mathcal{O} \) may be constructed? It is not clear how, or if, the work in the present article can be extended to this case. Going further, there is also the question in higher dimensions: given a closed and bounded set \( C \subseteq \mathbb{R}^n \) and an open cover \( \mathcal{O} \) of \( C \), can one describe a constructive process by which the smallest subcover of \( \mathcal{O} \) may be found? One might begin by looking at the \( n \)-cube \( C = [a_1, b_1] \times \cdots \times [a_n, b_n] \). Here one no longer has the ordering of the real numbers, so it’s not obvious at all where to begin.

For the present, then, we content ourselves with presenting the two counterexamples promised above. In the first case, consider the interval \( [0, 6] \) with the open cover

\[
\mathcal{O} = \{ O_1 = (-1, 3) \cup (5, 7), O_2 = (-1, 2) \cup (4, 7), O_3 = (1, 5) \},
\]

which may be seen visually in the following figure.

![Diagram of open cover](image)

Clearly the open cover \( \mathcal{O} \) does not just consist of intervals. We see from inspection that \( \{ O_2, O_3 \} \) is the smallest subcover of \( \mathcal{O} \). However, after first decomposing the intervals as described at the beginning of the previous section, the algorithms will work with the cover

\[
\mathcal{O}' = \{ (-1, 3), (5, 7), (-1, 2), (4, 7), (1, 5) \}.
\]

The greedy strategy will yield \((-1, 3), (1, 5) \) and \((4, 7) \) as the optimal subcover of \( \mathcal{O}' \). This corresponds to choosing the subcover \( \{ O_1, O_2, O_3 \} \) of \( \mathcal{O} \), which is not optimal.

Finally we discuss the generalization of Theorem 1 that states that any closed and bounded set \( C \subseteq \mathbb{R} \) is compact. Here is how this generalization may be proved constructively using our proof of Theorem 1. Because \( C \) is bounded it will be a subset of some \([a, b]\); we may assume that \( a \) is the greatest lower bound of \( C \) and that \( b \) is the least upper bound. Any open cover \( \mathcal{O} \) can be extended to an open cover \( \mathcal{O}' \) of \([a, b]\) by appending the open set \((a, b)\)\(\backslash C\). Having found a subcover of \( \mathcal{O}' \), one removes the open set \((a, b)\)\(\backslash C\) (if necessary) to obtain a subcover of \( \mathcal{O} \).

This process proves that \( C \) is compact, but the following example shows that the subcover found may not be the smallest possible. Consider the set \( C = [0, 1] \cup [3, 5] \) and the open cover

\[
\mathcal{O} = \{ O_1 = (-1, 2), O_2 = (1, 4), O_3 = (2, 6) \},
\]

which is visualised below.

![Diagram of open cover](image)
The smallest possible subcover consists of $O_1$ and $O_3$. However in the process described above, the set $(0, 5) \setminus C = (1, 3)$ is appended to the open cover to yield an open cover of $[0, 5]$. The algorithms choose $O_1$, $O_2$ and $O_3$ as the optimal subcover of $[0, 5]$. The set $(1, 3)$ is not in this subcover, so $\{O_1, O_2, O_3\}$ is returned as the subcover of $\emptyset$ for $C$. This subcover is evidently not optimal.

References


