Long-Time Behavior of Scalar Viscous Shock Fronts in Two Dimensions

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Abstract

We prove nonlinear stability in L^1 of planar shock front solutions to a viscous conservation law in two spatial dimensions, and obtain an expression for the asymptotic form of small perturbations. The leading-order behavior is shown rigorously to be governed by an effective diffusion coefficient depending on forces transverse to the shock front. The proof is based on a spectral analysis of the linearized problem.

1. Introduction

This paper concerns the scalar viscous conservation law

$$u_t + f(u)_x + g(u)_y = u_{xx} + u_{yy}, \tag{1.1}$$

where $f(u) = \frac{1}{2}u^2$. A planar viscous shock wave is a solution of (1.1) of the form $u = \varphi(x - st)$, where φ satisfies

(1.2)

$$-s\varphi' + f(\varphi)' = \varphi'' \ , \quad \varphi(x) \to u_l \text{ as } x \to -\infty, \quad \varphi(x) \to u_r \text{ as } x \to +\infty.$$

It is well known that such a solution exists for f satisfying an entropy condition (in particular, for any convex f), and that the solution φ must satisfy the Rankine-Hugoniot condition $(u_l - u_r)s = f(u_l) - f(u_r)$ [3]. We are concerned with the long-time behavior, in $L^1(\mathbb{R}^2)$, of solutions whose initial data is close to a planar shock wave.

We find that this long-time behavior is determined by an effective diffusion equation for the front motion itself. Specifically, for t large, $u(x,y,t)$ takes the form

$$u(x,y,t) \sim \varphi(x-\delta(y,t)-st), \tag{1.3}$$

where δ satisfies

$$\delta_t = \tau_2 \delta_{yy}. \tag{1.4}$$

The effective diffusion coefficient τ_2 is given by a formal perturbation calculation. This provides another example of a front whose large-scale behavior is determined by an effective front equation (compare the Cahn-Hilliard equation [1], [2], [10]).

The L^2 stability of shock solutions to (1.1) was shown in [5], using energy techniques like those that have been applied to systems of viscous conservation laws in one dimension ([7], [9], [4], [12]). However, energy estimates do not capture the long-time behavior described above; a formal calculation presented in [5] suggests the more precise description (1.3), (1.4). Writing a perturbation $v(x,y,t)$ of a stationary shock front as $v = e^{\tau t} e^{iky} \hat{v}(x)$, Goodman found the decay rate τ to be given to highest order (formally) as $\tau \sim -\tau_2 k^2$, where

$$\begin{aligned} -\tau_2 = -1 - \frac{\int_{-\infty}^{\infty} g(\varphi(x))^2/\varphi'(x)\, dx}{\int_{-\infty}^{\infty} \varphi'(x)\, dx} &\leqq -1 \\ &= -1 \iff g \equiv 0. \end{aligned} \tag{1.5}$$

Thus, forces (g) transverse to the direction of propagation of the shock ought to enhance the rate of decay of a perturbation, an effect analogous to Taylor dispersion in fluid mechanics (see for example [13], [14]). We will use pointwise estimates on Green's functions to verify the presence of this phenomenon.

As in [5] and [4], we begin by assuming a stationary shock front ($s = 0$) and writing

$$u(x, y, t) = \varphi(x - \delta(y, t)) + U_x(x, y, t), \tag{1.6}$$

where the shift $\delta(y, t)$ is chosen to make $U_x \in L^2$. Thus,

$$\int u(x, y, t) - \varphi(x - \delta(y, t))\, dx = 0. \tag{1.7}$$

The reason for this ansatz is that better energy estimates are available for the "integrated equation" for U than for the original equation for U_x.

As was noted in [5], a formal calculation suggests that the shift δ evolves according to equation (1.4) with effective diffusion coefficient τ_2. We verify this rigorously in L^1 for weak shocks, and show in addition that U_x decays in L^1 as $t \to \infty$. The long-time behavior of the perturbed shock front u can then be read from the expansion $u(x, y, t) = \varphi(x) - \delta\varphi'(x) + \mathcal{O}(\delta^2) + U_x$.

Remark 1.1. The choice of the L^1 norm is motivated by considering solutions of two one-dimensional diffusion equations

$$\partial_t u_1 = \nu_1 \partial_x^2 u_1,$$

$$\partial_t u_2 = \nu_2 \partial_x^2 u_2.$$

It is easy to verify that $\|u_1(t) - u_2(t)\|_{L^p} \to 0$ as $t \to \infty$ whenever $p > 1$, while $\|u_1(t) - u_2(t)\|_{L^1} \to 0$ only if $\int u_1(x,0)\,dx = \int u_2(x,0)\,dx$ and either $\nu_1 = \nu_2$ or $\int u_1(x,0)\,dx = 0$. Thus, L^1 is the only L^p norm strong enough to identify the correct diffusion coefficient.

For technical reasons, our proof requires use of the weighted norms L^p_B and H^s_B, defined by

$$\begin{aligned} \|\psi(x,y)\|_{L^p_B} &= \|e^{B(x)}\psi(x,y)\|_{L^p}, \\ \|\psi(x,y)\|_{H^s_B} &= \|e^{B(x)}\psi(x,y)\|_{H^s}, \end{aligned} \tag{1.8}$$

where

$$e^B(x) = \exp[-\tfrac{1}{2}\int_0^x f'(\varphi(s))\,ds]. \tag{1.9}$$

We also assume that

$$g(\varphi(x)) \in L^2(\mathbb{R}). \tag{1.10}$$

As noted in [5], equation (1.1) can always be normalized to make (1.10) true, and (1.10) implies that

$$|g(\varphi(x))| \leqq c(u_l - u_r)^2 \cdot \begin{cases} (u_l - \varphi(x)) & \text{if } x \leqq 0, \\ (\varphi(x) - u_r) & \text{if } x \geqq 0. \end{cases} \tag{1.11}$$

A formal statement of the result is as follows:

THEOREM 1.2. *Suppose that $\varphi(x)$ is a stationary ($s = 0$) shock solution of (1.1) with $f(u) = \frac{1}{2}u^2$, so that $\varphi \to \pm u_l$ as $x \to \mp\infty$. Suppose also that (1.10) holds. Let $u(x,y,t) = \varphi(x - \delta(y,t)) + U_x(x,y,t)$ be a solution of*

(1.1), as in (1.6), and let $\tilde{\delta}(y,t)$ denote the solution of (1.4) with initial data $\tilde{\delta}(\cdot,0) = \delta(\cdot,0)$. If $\|g'\|_{L^\infty}$, $\|g''\|_{L^\infty}$ and u_l are sufficiently small, and if the initial data $\|U\|_{H^1_B}$, $\|U\|_{L^1_B}$, $\|U_x\|_{L^1_B}$, $\|\delta\|_{H^1}$, $\|\delta\|_{L^1}$ are small as well, then

$$u(x,y,t) = \varphi(x) - \tilde{\delta}(y,t)\varphi'(x) + \mathcal{O}(t^{-1/2})$$

in $L^1(\mathbb{R}^2)$ as $t \to \infty$.

The ansatz (1.6), (1.7) leads to the equations

$$\begin{pmatrix} U \\ \delta \end{pmatrix}_t = A \begin{pmatrix} U \\ \delta \end{pmatrix} + NL(U,\delta), \tag{1.12}$$

where

(1.13)

$$A\begin{pmatrix} U \\ \delta \end{pmatrix} = \begin{pmatrix} U_{xx} + U_{yy} - f'(\varphi(x))U_x - (g'(\varphi(x))U)_y - V_y + g(\varphi(x))\delta_y \\ \delta_{yy} + \alpha_y \end{pmatrix},$$

$$V(x,y,t) = \begin{cases} -\int_{-\infty}^x g''(\varphi(s))\varphi'(s)U(s)\,ds - \alpha(y,t)\,[\varphi(x)-1] & (x<0); \\ \int_x^\infty g''(\varphi(s))\varphi'(s)U(s)\,ds - \alpha(y,t)\,[\varphi(x)+1] & (x>0)\,, \end{cases}$$

$$\alpha(y,t) = \frac{-1}{\int \varphi'\,dx}\int g''(\varphi(x))\varphi'(x)U(x,y,t)\,dx,$$

and the nonlinear terms $NL(U,\delta)$ arise both from the terms f and g in (1.1) and from a centering of $\varphi(x-\delta)$ around $\varphi(x)$ (see §4). The proof of Theorem 1.2 is based on an analysis of the linearized perturbation equations.

The analysis begins with applying the exponential weight (1.9) to equation (1.12) and taking the Fourier transform of the linear part. This yields a family of problems in one spatial dimension, parametrized by k, where k is the wave number in the y-direction. Most of the work of proving Theorem

1.2 lies in finding pointwise estimates for the solution operators associated with the linear operators $\widehat{A}(k)$, by means of a spectral analysis. The effective diffusion coefficient τ_2 turns out to be determined by the behavior of an eigenvalue of $\widehat{A}(k)$, for k near zero. Contributions from the continuous spectrum are treated in the spirit of [11] and [8]. The nonlinear terms in (1.12) can be handled by a comparatively simple variation-of-parameters argument once linear estimates are obtained.

2. Linear spectral theory

To set up the spectral problem, we first conjugate the linearized version of (1.12) by

$$\begin{pmatrix} e^B(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where $e^B(x)$ is as in (1.9). Applying a Fourier transform in the y-direction then leads to the equation

$$\begin{pmatrix} \widehat{U}(x,k,t) \\ \hat{\delta}(k,t) \end{pmatrix}_t = \widehat{A}(k) \begin{pmatrix} \widehat{U}(x,k,t) \\ \hat{\delta}(k,t) \end{pmatrix} ,$$

where $\widehat{U}$ is the Fourier transform of $e^{B(x)}U$, $\hat{\delta}$ is the Fourier transform of δ, and

$$\begin{aligned} \widehat{A}(k) \begin{pmatrix} v(x,k,t) \\ \delta(k,t) \end{pmatrix} &= \\ &= \begin{pmatrix} \widehat{M}(k)v & -ike^{B(x)}g(\varphi)\delta \\ -\frac{ik}{2u_l}\int_{-\infty}^{\infty} g''(\varphi)\varphi'(s)e^{-B(s)}v(s)\,ds & -k^2\delta \end{pmatrix}, \end{aligned}$$

$$\widehat{M}(k)v = \left[\partial_x^2 - \tfrac{1}{4}f'(\varphi)^2 - k^2 + \tfrac{1}{2}f''(\varphi)\varphi' + ikg'(\varphi)\right] v + ik\widehat{V}(v)(x,k,t),$$

$$\widehat{V}(v) = e^{B(x)} \begin{cases} -\int_{-\infty}^{x} g''(\varphi)\varphi' e^{-B} v(s)\, ds \\ \quad + \dfrac{[\varphi(x) - u_l]}{-2u_l)} \int_{-\infty}^{\infty} g''(\varphi)\varphi' e^{-b} v(s)\, ds & (x < 0) \\ \int_{x}^{\infty} g''(\varphi)\varphi' e^{-B} v(s)\, ds \\ \quad + \dfrac{[-u_l - \varphi(x)]}{2u_l} \int_{-\infty}^{\infty} g''(\varphi)\varphi' e^{-B} v(s)\, ds & (x > 0). \end{cases}$$

Note: The ˆ symbol on $\widehat{U}$, $\hat{\delta}$ etc. will occasionally be omitted when this causes no confusion.

We will see that each operator $\widehat{A}(k)$ has essential spectrum confined to a sector in the left half plane. Because of the exponential weight, in fact, we have $\sigma_{ess}(\widehat{A}(k)) \subset \{\mathrm{Re}\,\lambda \leqq \frac{1}{4}u_l^2\}$ for all k. In addition, for k near zero $\widehat{A}(k)$ has a single, simple eigenvalue $\mu(k)$, satisfying $\mu(0) = 0$. This eigenvalue $\mu(k)$ will control the long–time behavior of solutions to the linearized version of the full problem (1.12).

The following lemma describes the behavior of $\mu(k)$, and of the associated eigenfunctions, for k near zero.

Lemma 2.1. *There exist $k_0 > 0$ and functions $\mu(k) \in \mathbb{R}$, $\eta(k) \in \mathbb{R}$, $\tilde{w}(x,k) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $w(x,k) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, defined and analytic for $|k| \leqq k_0$, such that $\tilde{w}(x,k)$ and $w(x,k)$ are exponentially decaying in space* *and*

$$\begin{aligned}
&\mu(k) \text{ is a simple eigenvalue of } \widehat{A}(k) \text{ with } \mu(0) = 0,\\
&\sigma(\widehat{A}(k)) \cap \{Re\,\lambda \geqq -\tfrac{1}{8}u_l^2\} = \{\mu(k)\},\\
&\widehat{A}(k)\begin{pmatrix} w(k) \\ 1 \end{pmatrix} = \mu(k)\begin{pmatrix} w(k) \\ 1 \end{pmatrix},\\
&\begin{pmatrix} \tilde{w}(k) \\ \eta(k) \end{pmatrix}^T \widehat{A}^*(k) = \mu(k)\begin{pmatrix} \tilde{w}(k) \\ \eta(k) \end{pmatrix}^T,\\
&(x,\eta)\cdot(w,1) \equiv \int \bar{w}(x,k)\tilde{w}(x,k)\,dx + \eta(k) = 1.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mu(k) = -\tau_2 k^2 + \mathcal{O}(k^3),\\
&w(k) = \mathcal{O}(k^m),\\
&\tilde{w} = \mathcal{O}(k^n),\\
&\eta = 1 + \mathcal{O}(k^2)
\end{aligned}$$

for some $m, n \geqq 1$ *as* $k \to 0$*, where* τ_2 *is as in (1.5).*

Proof: When $k = 0$, the operator $\widehat{A}(0)$ is simply $\begin{pmatrix} \widehat{M}(0) & 0 \\ 0 & 0 \end{pmatrix}$. Thus, 0 is a simple eigenvalue of $\widehat{A}(0)$, and $\sigma(\widehat{A}(0))\backslash\{0\} \subset \{\mathrm{Re}\,\lambda \leqq -u_l^2/4\}$. By standard results of spectral theory [6], $\mu(k)$, $w(k)$, $\tilde{w}(k)$ and $\eta(k)$ are defined and analytic in a neighborhood of $k = 0$. Furthermore, $\mu(k)$ is a simple eigenvalue of $\widehat{A}(k)$, and since $\widehat{A}(0)$ is selfadjoint, we have

$$\begin{pmatrix} w(0) \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{w}(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By analyticity, we may write $w(k) = kw_1(k)$, $\tilde{w}(k) = k\tilde{w}_1(k)$, where w_1 and $\tilde{w}_1$ are analytic functions of k with values in $L^2(\mathbb{R})$. The condition that $(\tilde{w}, \eta)\cdot(w,1) = 1$ then implies that $\eta(k) = 1 - k^2\int \bar{w}_1\tilde{w}_1\,dx$.

The proof that τ_2 has the desired form is as in [5]; it depends on a Taylor expansion and noting that

$$-\tau_2 + 1 = \frac{\int g'(\varphi)\hat{v}_1 \, dx}{\int \varphi' \, dx},$$

where $\partial_x \hat{v}_1 - f'(\varphi)\hat{v}_1 = g(\varphi)$.

The fact that $\sigma(\widehat{A}(k)) \cap \{\operatorname{Re}\lambda \geqq -\frac{1}{8}u_l^2\} = \{\mu(k)\}$ is a consequence of resolvent estimates to follow (Lemma 2.3).

We now look for bounds on the continuous spectral part of $\widehat{A}(k)$.

Let $\widehat{M_0}(k) = \left[\partial_x^2 - \frac{1}{4}f'(\varphi)^2 - k^2 + \frac{1}{2}f''(\varphi)\varphi'\right] v$, so that

$$\widehat{M}(k)v = \widehat{M_0}(k)v + ik[g'(\varphi)v + \widehat{V}(v)].$$

Also, let

$$\gamma(k, \lambda) = \sqrt{\tfrac{1}{4}u_l^2 + \lambda + k^2}$$

and let $\widehat{H}(x, w, k, \lambda)$ be the resolvent Green's function for $\widehat{M}(k)$; that is, let the solution of

$$(\widehat{M}(k) - \lambda)v = h(x)$$

be given by

$$v(x) = \int \widehat{H}(x, w, k, \lambda) h(w) \, dw.$$

The following lemma states pointwise bounds on $\widehat{H}$, as a first step towards finding bounds on the resolvent for $\widehat{A}(k)$.

Lemma 2.2. *a)* *Suppose that $k \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and that (k, λ) are such that*

$$Re\,\gamma \geqq \tfrac{1}{2} u_l + \epsilon \tag{2.1}$$

for some $\epsilon > 0$. If $\|g'\|_{L^\infty}$, $\|g''\|_{L^\infty}$ are sufficiently small, then there exists $C > 0$ depending only on ϵ and u_l such that

$$|\widehat{H}(x, w, k, \lambda)| \leqq C \frac{1}{|\gamma|} e^{-\frac{1}{2}(u_l+\epsilon)|x-w|}$$

and

$$|\partial_x \widehat{H}(x, w, k, \lambda)| \leqq C e^{-\frac{1}{2}(u_l+\epsilon)|x-w|}.$$

b) *If $Re\,\gamma \ngeqq \frac{1}{2} u_l^2 + \epsilon$, but instead (k, λ) are such that*

$$Re\,\gamma \geqq a$$

for some $a > 0$, then the conclusion of part a) holds with $u_l + \epsilon$ replaced by a.

Proof: We give the proof only for part a). First, we state estimates on the resolvent Green's function for $\widehat{M_0}(k)$, which we denote by $\widehat{H}_0(x, w, k, \lambda)$. As in Sattinger [11], the following pointwise bounds hold:

$$\left|\widehat{H}_0(x, w, k, \lambda)\right| \leqq \frac{1}{2|\gamma|} \left|e^{-\gamma|x-w|}\right| \left(1 + \mathcal{O}\left(\frac{1}{|\gamma|}\right)\right);$$
$$\left|\partial_x \widehat{H}_0(x, w, k, \lambda)\right| \leqq \frac{1}{2} \left|e^{-\gamma|x-w|}\right| \left(1 + \mathcal{O}\left(\frac{1}{|\gamma|}\right)\right).$$

These bounds are obtained by examining a more or less explicit expression for $\widehat{H}_0$ as a solution of $\widehat{M_0}\widehat{H}_0 = \delta(x)$. Note that the error $1/|\gamma|$ is uniformly bounded for (k, λ) satisfying the hypotheses of the lemma.

Continuing, we write the full scalar Green's function $\widehat{H}$ as a Neumann series involving $\widehat{H}_0$:

$$\widehat{H}(x,w,k,\lambda) = \sum_{n=0}^{\infty} (-ik)^n \widehat{H}_n(x,w,k,\lambda),$$

where $\widehat{H}_0$ is as previously defined and

$$\begin{aligned} \widehat{H}_{n+1}(x,w,k,\lambda) = \int & \widehat{H}_0(x,z,k,\lambda) \cdot \\ & \cdot \left[g'\left(\varphi(z)\right) \widehat{H}_n(z,w,k,\lambda) + \widehat{V}(\widehat{H}_n)(z,w,k,\lambda) \right] \, dz \end{aligned}$$

for $n \geqq 0$.

We find pointwise estimates for $\widehat{H}_1$ in several steps, as a model for estimates on all the $\widehat{H}_n$. First,

$$\left| g'\left(\varphi(z)\right) \widehat{H}_0(z,w,k,\lambda) \right| \leqq C \|g'\|_{L^\infty} \frac{1}{2|\gamma|} \left| e^{-\gamma|z-w|} \right|, \tag{2.2}$$

where the constant depends on r for $|\gamma| \geqq r$. Next, if $z > 0$ then

$$\begin{aligned} &\left| e^{B(z)} \int_z^\infty g''(\varphi(s))\varphi'(s) e^{-B(s)} \widehat{H}_0(s,w,k,\lambda) \, ds \right| \leqq \\ &\quad \leqq C \frac{1}{|\gamma|} \|g''\|_{L^\infty} e^{\frac{1}{2}u_l|z|} \int_z^\infty e^{-\frac{1}{2}u_l|s|} \, ds \sup_{z<s<\infty} \left(e^{-u_l|s|} e^{-\mathrm{Re}\,\gamma|s-w|} \right) \\ &\quad \leqq C \frac{1}{|\gamma|} \|g''\|_{L^\infty} e^{-\frac{1}{2}(u_l+\epsilon)|z-w|}. \end{aligned} \tag{2.3}$$

The estimate for $z < 0$ is similar.

Finally,

$$\begin{aligned} &\left| e^{B(z)} \left[\frac{u_l - \varphi(z)}{2u_l} \right] \int_{-\infty}^\infty g''(\varphi(s))\varphi'(s) e^{-B(s)} \widehat{H}_0(s,w) \, ds \right| \leqq \\ &\qquad \leqq C \frac{1}{|\gamma|} \|g''\|_{L^\infty} e^{-\frac{1}{2}u_l|z|} \int_{-\infty}^\infty e^{-3u_l|s|/2} e^{-\gamma|w-s|} \, ds \\ &\qquad \leqq C \frac{1}{|\gamma|} \|g''\|_{L^\infty} e^{-\gamma|z-w|}. \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) gives

$$\left|g'\left(\varphi(z)\right)\widehat{H}_0(z,w,k,\lambda)+\widehat{V}(\widehat{H}_0)(z,w,k,\lambda)\right| \leqq$$
$$\leqq C\frac{1}{|\gamma|}\left(\|g'\|_{L^\infty}+\|g''\|_{L^\infty}\right)e^{-\frac{1}{2}(u_l+\epsilon)|z-w|}.$$

With (2.1), it follows that

$$\int \widehat{H}_0(x,z,k,\lambda)\left[g'\left(\varphi(z)\right)\widehat{H}_0(z,w,k,\lambda)+\widehat{V}(\widehat{H}_0)(z,w,k,\lambda)\right]\,dz \leqq$$
$$\leqq C_1\frac{1}{|\gamma|^2}\left(\|g'\|_{L^\infty}+\|g''\|_{L^\infty}\right)e^{-\frac{1}{2}(u_l+\epsilon)|x-w|}.$$

Here the constant C_1 is independent of (k,λ), provided that (k,λ) satisfies (2.1).

We can choose $\|g'\|_{L^\infty}$, $\|g''\|_{L^\infty}$ so small that $C_1\left(\|g'\|_{L^\infty}\|g''\|_{L^\infty}\right) < |\gamma|/|k|$ whenever (k,λ) satisfies (2.1). Thus, we can iterate the above estimate to obtain

$$\begin{aligned}
\left|\widehat{H}(x,w,k,\lambda)\right| &\leqq \sum_{n=0}^{\infty}|k|^n\left|\widehat{H}_n(x,w,k,\lambda)\right| \\
&\leqq \sum_{n=0}^{\infty}\left[\frac{C_1|k|}{|\gamma|u_l}\left(\|g'\|_{L^\infty}+\|g''\|_{L^\infty}\right)\right]^n\cdot\frac{C}{|\gamma|}e^{-\frac{1}{2}(u_l+\epsilon)|x-w|} \\
&\leqq \frac{C}{|\gamma|}e^{-\frac{1}{2}(u_l+\epsilon)|x-w|}.
\end{aligned}$$

The analogous estimate for $\partial_x\widehat{H}(x,w,k,\lambda)$ is proved similarly.

We now seek pointwise estimates on the Green's function which is the inverse of

$$\lambda-\widehat{A}(k)=\begin{pmatrix}\lambda-\widehat{M}(k) & ike^{B}g(\varphi)\\ \frac{-ik}{2u_l}\int\cdot g''(\varphi)\varphi' e^{-B} & \lambda+k^2\end{pmatrix}=\lambda-(D-ikS),$$

where

$$D(k) = \begin{pmatrix} \widehat{M}(k) & 0 \\ 0 & -k^2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & e^B g(\varphi) \\ -\frac{1}{2u_l} \int \cdot g''(\varphi)\varphi' e^{-B} & 0 \end{pmatrix}.$$

We write this Green's function as

(2.5)
$$\left(\lambda - \widehat{A}(k)\right)^{-1} \begin{pmatrix} v(x) \\ \delta \end{pmatrix} = \begin{pmatrix} \int \widehat{H}_{11}(x,w,k,\lambda)v(w)\,dw + \widehat{H}_{12}(x,k,\lambda)\delta \\ \int \widehat{H}_{21}(w,k,\lambda)v(w)\,dw + \widehat{H}_{22}(k,\lambda)\delta \end{pmatrix}.$$

Lemma 2.3. *a)* *Suppose that $k \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and that (k,λ) satisfies the following conditions for some $\epsilon > 0$:*

$$Re\,\gamma \geqq \tfrac{1}{2}u_l + \epsilon,$$

$$|\lambda + k^2| \geqq \epsilon,$$

$$\left|\frac{k^2}{\lambda + k^2}\right| \leqq 10^6.$$

If $\|g''\|_{L^\infty}$ is sufficiently small and if (1.10) holds, then for some constant $C > 0$ depending on u_l, ϵ and $\|g'\|_{L^\infty} + \|g''\|_{L^\infty}$, the following estimates hold:

$$|\widehat{H}_{11}(x,w,k,\lambda)| \leqq C\frac{1}{|\gamma|}e^{-\frac{1}{2}u_l|x-w|},$$

$$|\widehat{H}_{12}(x,k,\lambda)| \leqq C\frac{1}{|\gamma|}\left|\frac{k}{\lambda + k^2}\right| e^{-\frac{1}{2}u_l|x|},$$

$$|\widehat{H}_{21}(w,k,\lambda)| \leqq C\frac{1}{|\gamma|}\left|\frac{k}{\lambda + k^2}\right| e^{-\frac{1}{2}u_l|w|},$$

$$|\widehat{H}_{22}(k,\lambda)| \leqq \left(1 + \frac{C}{|\gamma|}\right)\left|\frac{1}{\lambda + k^2}\right|,$$

$$|\partial_x \widehat{H}_{11}(x,w,k,\lambda)| \leqq Ce^{-\frac{1}{2}u_l|x-w|},$$

$$|\partial_x \widehat{H}_{12}(x,k,\lambda)| \leqq C\left|\frac{k}{\lambda + k^2}\right| e^{-\frac{1}{2}u_l|x|}.$$

b) *Suppose that $k \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and that (k, λ) satisfies the following conditions for some $a > 0$,$\epsilon > 0$:*

$$
\begin{aligned}
&Re\,\gamma \geqq a, \\
&|\lambda + k^2| \geqq \epsilon, \qquad (2.6)\\
&\left| \frac{k^2}{\lambda + k^2} \right| \leqq 10^6.
\end{aligned}
$$

If $\|g''\|_{L^\infty}$ is sufficiently small and if (1.10) holds, then the estimates in part a) hold with u_l replaced by a, and with the constant C depending on a.

Proof: Again, we write $(\lambda - \widehat{A}(k))^{-1}$ formally as a Neumann series:

$$
\begin{aligned}
(\lambda - D + ikS)^{-1} &= \left(I - ik(D-\lambda)^{-1}S\right)^{-1}(\lambda - D)^{-1} \\
&= \left[\sum_{n=0}^{\infty} \left(ik(D-\lambda)^{-1}S\right)^n\right](\lambda - D)^{-1}.
\end{aligned}
$$

In general, if T is an operator on $L^p(\mathbb{R}) \times \mathbb{R}$ defined by

$$
T \begin{pmatrix} v(x) \\ \delta \end{pmatrix} = \begin{pmatrix} \delta f(x) \\ \int vh(x)\, dx \end{pmatrix}
$$

then

$$
\sum_{n=1}^{\infty} T^n \begin{pmatrix} v \\ \delta \end{pmatrix} = \left(1 - \int fh\right)^{-1} \begin{pmatrix} f(x)\left[\int vh + \delta\right] \\ \int vh + \delta \int fh \end{pmatrix},
$$

provided $\int fh < 1$. In the case at hand, with $T = ik(\lambda - D)^{-1}S$, we have

$$
\begin{aligned}
f(x) &= ik\left(\widehat{M}(k) - \lambda\right)^{-1}\left(e^B g(\varphi)\right) \\
&= ik \int \widehat{H}(x, w, k, \lambda) g(\varphi) e^{B(w)}\, dw
\end{aligned}
$$

and

$$
h(x) = \frac{ik}{\lambda + k^2}\left(\frac{-1}{2u_l}\right) g''(\varphi)\varphi' e^{-B(x)},
$$

so that

$$\left|\int fh\right| \leqq$$
$$\leqq C\left|\frac{k^2}{\lambda+k^2}\right|\frac{1}{|\gamma|}\|g''\|_{L^\infty}\int e^{-3u_l|x|/2}\int e^{-\frac{1}{2}u_l|x-w|}\,|g(\varphi(w))|\,e^{\frac{1}{2}u_l|w|}\,dw\,dx.$$

But (1.11) implies that $|g(\varphi(w))| \leqq Cu_l|\varphi'(w)|$, so

$$\left|\int fh\right| \leqq C\left|\frac{k^2}{\lambda+k^2}\right|\frac{1}{|\gamma|}\|g''\|_{L^\infty}.$$

We can choose $\|g''\|_{L^\infty}$ small enough that $|\int fh| < 1/2$ whenever (k,λ) satisfies the hypotheses of the lemma. The following representations for the components of $\left(\widehat{A}(k)-\lambda\right)^{-1}$, obtained from the formal series, are then well-defined:

$$\widehat{H}_{11}(x,w,k,\lambda) = \widehat{H}(x,w,k,\lambda) + \left[1-\int fh\right]^{-1} f(x)\int h(z)\widehat{H}(z,w,k,\lambda)\,dz,$$
$$\widehat{H}_{12}(x,k,\lambda) = \left[1-\int fh\right]^{-1} f(x)\frac{1}{\lambda+k^2},$$
$$\widehat{H}_{21}(w,k,\lambda) = \left[1-\int fh\right]^{-1}\int h(z)\widehat{H}(z,w,k,\lambda)\,dz,$$
$$\widehat{H}_{22}(k,\lambda) = \frac{1}{\lambda+k^2} + \left[1-\int fh\right]^{-1}\int fh\frac{1}{\lambda+k^2}.$$

We have

$$\left|\int h(z)\widehat{H}(z,w)\,dz\right| \leqq \frac{C}{u_l}\left|\frac{k}{\lambda+k^2}\right|\frac{1}{|\gamma|}\|g''\|_{L^\infty}\int e^{-3u_l|z|/2}e^{-\frac{1}{2}u_l|z-w|}\,dz$$

and

$$|f(x)| \leqq Cu_l\frac{k}{|\gamma|}\int e^{-\frac{1}{2}(u_l+\epsilon)|x-w|}e^{-\frac{1}{2}u_l|w|}\,dx,$$

provided that (1.10) holds. Combining these bounds with the bounds on $|\widehat{H}|$ from Lemma 2.2 gives the desired estimates on $|\hat{H}_{ij}|$ $(j=1,2)$; estimates on $|\partial_x \hat{H}_{ij}|$ are proved similarly.

3. The linear evolution operator

With estimates in hand for the resolvent Green's functions associated with each operator $\widehat{A}(k)$, we now can obtain estimates for the evolution operators $e^{\widehat{A}(k)t}$. These estimates are obtained from the Cauchy integral formula, following Sattinger [11]. We present the estimates in two statements, one for $|k| \geqq k_0 > 0$ and one for $|k| \leqq k_0$. Recall that k_0 is a radius within which the Taylor expansion for the eigenvalue $\mu(k)$ is valid (Lemma 2.1).

A few definitions are required. First, for $|k| \leqq k_0$ define $P(k)$ to be the projection onto the $\mu(k)$ eigenspace of $\widehat{A}(k)$. That is,

$$P(k)\begin{pmatrix} v(x) \\ \delta \end{pmatrix} = \left(\int \tilde{w}(k) v \, dx + \eta(k)\delta\right)\begin{pmatrix} w(k) \\ 1 \end{pmatrix}, \tag{3.1}$$

where $\tilde{w}$, η, w are as in Lemma 2.1. Next, fix an angle $\theta \in (\pi/2, 3\pi/4]$ such that $|\arg\lambda| > \theta$ for all $\lambda \in \sigma(\widehat{A}(k))\backslash\{0\}$, $k \in \mathbb{R}$ (such a θ exists by Lemma 2.3). For $|k| \geqq k_0$, define a contour of integration Γ_k by

$$\Gamma_k = \left\{-k^2/2 + \rho e^{\pm i\theta} \;:\; \rho \geqq 0\right\},$$

and for $\omega > 0$ to be chosen later, define

$$\Gamma_0 = \left\{-\omega + \rho e^{\pm i\theta} \;:\; \rho \geqq 0\right\}. \tag{3.2}$$

We will denote $e^{\widehat{A}(k)t}$ by $\widehat{G}$, so that

$$\widehat{G}_{ij} = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_k} e^{\lambda t}\widehat{H}_{ij}\, d\lambda & \text{if } |k| > k_0, \\ \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\widehat{H}_{ij}\, d\lambda + e^{\mu(k)t}P(k)_{ij} & \text{if } |k| \leqq k_0, \end{cases}$$

where $\widehat{H}_{ij}$, $j = 1, 2$ are as in (2.5). Finally, we define

$$Q\widehat{G}_{ij} = \int_{\Gamma_0} e^{\lambda t}\widehat{H}_{ij}\, d\lambda$$

and

$$G_{ij} = \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{iky}\widehat{G}_{ij}\, dk. \tag{3.3}$$

Lemma 3.1. *Suppose that $k \in \mathbb{C}$ and $|k| \geqq k_0$. Then there exists $C > 0$ depending only on k_0, θ, u_l, and the constant from Lemma 2.3, such that*

$$\left|\widehat{G}_{11}(x, w, k, t)\right| \leqq \frac{C}{u_l}\frac{1}{\sqrt{t}}e^{-k^2t/2}e^{-\frac{1}{2}u_l|w-x|},$$

$$\left|\widehat{G}_{12}(x, k, t)\right| \leqq \frac{C}{\sqrt{t}}e^{-k^2t/2}e^{-\frac{1}{2}u_l|x|},$$

$$\left|\widehat{G}_{21}(w, k, t)\right| \leqq \frac{C}{k^2}\frac{1}{\sqrt{t}}|k|e^{-k^2t/2}e^{-\frac{1}{2}u_l|w|} \leqq \frac{C}{k_0^2}\frac{1}{\sqrt{t}}|k|e^{-k^2t/2}e^{-\frac{1}{2}u_l|w|},$$

$$\left|\widehat{G}_{22}(k, t)\right| \leqq \frac{C}{k^2}\frac{1}{\sqrt{t}}e^{-k^2t/2} \leqq \frac{C}{k_0^2}\frac{1}{\sqrt{t}}e^{-k^2t/2},$$

$$\left|\partial_x\widehat{G}_{11}(x, w, k, t)\right| \leqq \frac{C}{u_l}\frac{1}{t}e^{-k^2t/2}e^{-\frac{1}{2}u_l|w-x|},$$

$$\left|\partial_x\widehat{G}_{12}(x, k, t)\right| \leqq \frac{C}{\sqrt{t}}e^{-k^2t/2}e^{-\frac{1}{2}u_l|x|}.$$

Proof: First, note that

$$\int_{\Gamma_k} e^{\lambda t}\widehat{H}_{ij}(\cdot, k, \lambda)\, d\lambda =$$

$$= \int_{\rho=0}^{\infty} e^{-k^2t/2}e^{\rho t(\cos\theta + i\sin\theta)}\widehat{H}_{ij}(\cdot, k, -k^2/2 + \rho e^{\pm i\theta})\, d\rho.$$

As in Sattinger [11], we have

$$\int_0^\infty e^{\rho t\cos\theta}\frac{1}{|\gamma|}\,d\rho \leqq \int_0^\infty \frac{e^{\rho t\cos\theta}}{\sqrt{(\rho\sin\theta)}}\,d\rho$$
$$\leqq \frac{C(\theta)}{\sqrt{t}}.$$

Also, for $\lambda \in \Gamma_k$ we have $|1/(\lambda+k^2)| \leqq |2/k^2| \leqq |2/k_0^2|$, and $c_1|\sqrt{\lambda+k^2}| \leqq \gamma \leqq c_2|\sqrt{\lambda+k^2}|$ for some c_1, $c_2 > 0$. Combining these observations with the estimates from Lemma 2.3 yields the desired result.

LEMMA 3.2. *Fix $k_0 > 0$, and suppose that $k \in \mathbf{C}$, $|k| \leqq k_0$. Assume that the constants k_0, θ, $-\omega$ in (3.2) are chosen so that Γ_0 lies between $\sigma_{ess}\widehat{A}(k)$ and the eigenvalue $\mu(k)$ for all $|k| \leqq k_0$ (this is possible by Lemmas 2.1 and 2.3. Also assume that the constant a in Lemma 2.2 is such that $Re\gamma > a$ whenever $\lambda \in \Gamma_0$. Then there exists $C > 0$ depending only on u_l, k_0, θ, ω and the constants from Lemma 2.3 b) such that*

$$\left|Q\widehat{G}_{11}\right| \leqq C\frac{1}{\sqrt{t}}e^{-\omega t}e^{-a|w-x|/2},$$
$$\left|Q\widehat{G}_{12}\right| \leqq \frac{C}{\sqrt{t}}e^{-\omega t}e^{-a|x|/2},$$
$$\left|Q\widehat{G}_{21}\right| \leqq \frac{C}{\sqrt{t}}e^{-\omega t}e^{-a|w|/2},$$
$$\left|Q\widehat{G}_{22}\right| \leqq \frac{C}{\sqrt{t}}e^{-\omega t},$$

$$\left|\partial_x Q\widehat{G}_{11}\right| \leqq C\frac{1}{t}e^{-\omega t}e^{-a|w-x|/2},$$
$$\left|\partial_x Q\widehat{G}_{12}\right| \leqq \frac{C}{\sqrt{t}}e^{-\omega t}e^{-a|x|/2}.$$

The proof of Lemma 3.2 is similar to the proof of Lemma 3.1.

We can now use the inverse Fourier transform to obtain pointwise estimates on the Green's function, G, for the evolution governed by the linear operator A. Specifically, applying the estimates in Lemmas 2.1, 2.3, 3.1 and 3.2 to the Fourier integral formula (3.3) leads to the following result.

LEMMA 3.3. *Let $w(x,k)$, $\tilde{w}(x,k)$ denote the eigenvector and adjoint eigenvector of $\widehat{A}(k)$ associated with the eigenvalue $\mu(k)$, as in Lemma 2.1. Suppose that $w(x,k) \sim k^m w_1(x)$ and $\tilde{w}(x,k) \sim k^n \tilde{w}_1(x)$ as $|k| \to 0$. Then as $t \to \infty$, for any $p \in [1,\infty]$, $\rho \in \{0,1\}$ we have*

$$\partial_y^\rho G_{11}(x,w,y,t) \sim w_1(x)\tilde{w}_1(w)\frac{1}{2\pi}\int e^{iky}e^{-\tau_2 k^2 t}k^{m+n}\,dk \quad \text{in } L^p(\mathbb{R}^3)$$
$$\partial_y^\rho G_{12}(x,y,t) \sim w_1(x)\frac{1}{2\pi}\int e^{iky}e^{-\tau_2 k^2 t}k^{m}\,dk \quad \text{in } L^p(\mathbb{R}^2)$$
$$\partial_y^\rho G_{21}(w,y,t) \sim \tilde{w}_1(w)\frac{1}{2\pi}\int e^{iky}e^{-\tau_2 k^2 t}k^{n}\,dk \quad \text{in } L^p(\mathbb{R}^2)$$
$$\partial_y^\rho G_{22}(y,t) \sim \frac{1}{2\pi}\int e^{iky}e^{-\tau_2 k^2 t}\,dk \quad \text{in } L^p(\mathbb{R}),$$
$$\partial_y\partial_x^\rho G_{11}(x,w,y,t) \sim \partial_x w_1(x)\tilde{w}_1(w)\frac{1}{2\pi}\int e^{iky}e^{-\tau_2 k^2 t}k^{m+n}\,dk \quad \text{in } L^p(\mathbb{R}^3)$$
$$\partial_y\partial_x^\rho G_{12}(x,y,t) \sim \partial_x w_1(x)\frac{1}{2\pi}\int e^{iky}e^{-\tau_2 k^2 t}k^{m}\,dk \quad \text{in } L^p(\mathbb{R}^2).$$

Proof: We present the proof only for G_{22}; the other cases are similar. A key observation is that the estimates in Lemmas 2.1, 2.3, 3.1 and 3.2 are uniform for complex k with $\operatorname{Im} k$ sufficiently small.

Using the estimates in Lemma 3.1, one can shift the contour of integration in (3.3):

$$G_{22} = \frac{1}{2\pi}\int_{\substack{z=k+i\eta \\ k\in\mathbb{R}}} e^{ikz}\widehat{G}_{22}(z)\,dz \tag{3.4}$$

provided that $|\eta|$ is sufficiently small and that $\eta y > 0$.

From the estimates in Lemmas 3.1 and 3.2, respectively, it follows that if $|\eta|$ and $k_0 > 0$ are sufficiently small, and if $\eta^2 < k_0^2$, then there exists $C > 0$ independent of η and t (for t bounded away from zero) such that

$$\left| \int\limits_{\substack{z = k + i\eta \\ |k| > k_0}} e^{izy} \widehat{G}_{22} \, dz \right| \leqq C e^{-\eta y} \frac{1}{\sqrt{t}} e^{-t[k_0^2 - \eta^2]/2}$$

and

$$\left| \int\limits_{\substack{z = k + i\eta \\ |k| \leqq k_0}} e^{izy} Q \widehat{G}_{22} \; dz \right| \leqq C e^{-\eta y} e^{-\omega t}. \tag{3.5}$$

This shows that the contributions to the contour integral (3.4) from "large" frequencies $|k| > k_0$, and from the continuous spectrum for frequencies k near zero, decay rapidly in time.

It remains to consider the contribution from the eigenvalue $\mu(k)$ for k near zero; that is, to estimate

$$\int\limits_{\substack{z = k + i\eta \\ |k| < k_0}} e^{izy} e^{\mu(z)t} P_{22}(z) \, dz = - \int\limits_{\substack{z = r(\pm k_0 + i\eta) \\ r \in [0,1]}} e^{izy} e^{\mu(z)t} P_{22}(z) \, dz, \tag{3.6}$$

where P is defined as in (3.1). We estimate this integral in the following three claims.

Note: For the rest of this proof, we assume that $\eta y > 0$, and that $\alpha > 0$ is chosen such that

$$1/2 > \alpha > 1/N, \quad \text{where } \mu(z) + \tau_2 z^2 = \mathcal{O}(z^N) \quad \text{as } z \to \infty.$$

Claim 1: There exists $C > 0$ independent of t (for t bounded away from zero) such that the following inequality holds:

$$\left| \int\limits_{\substack{z = r(\pm k_0 + i\eta) \\ r \in [t^{-\alpha}, 1]}} e^{izy} e^{\mu(z)t} P_{22}(z)\, dz \right| \leqq C t^{-1/2} e^{-\eta y t^{-\alpha}}.$$

Claim 2: There exists $C > 0$ independent of t (for t bounded away from zero) such that

$$\left| \int\limits_{\substack{z = r(\pm k_0 + i\eta) \\ r \in [0, t^{-\alpha}]}} e^{izy} e^{\mu(z)t} (P_{22}(z) - 1)\, dz \right| \leqq C t^{-1} \left(1 + y^2/t\right)^{-1}. \tag{3.7}$$

Claim 3:

$$\left| \int\limits_{\substack{z = r(\pm k_0 + i\eta) \\ r \in [0, t^{-\alpha}]}} e^{izy} \left[e^{\mu(z)t} - e^{-\tau_2 z^2 t} \right] dz \right| \leqq C \left(1 + y^2/t\right)^{-1} t^{1/2 - N\alpha}. \tag{3.8}$$

Proof of Claim 1: This follows from the facts that $P_{22}(z) = 1 + \mathcal{O}(z)$ and $\mu(z) = \mathcal{O}(z^2)$ for z near zero.

Proof of Claim 2: Writing $P_{22}(z) - 1 = zQ(z)$, where Q is analytic in a neighborhood of $z = 0$, and making the substitution $z = mt^{-1/2}$, we can write the integral on the left-hand side of (3.7) as

$$\frac{1}{t} \int\limits_{\substack{m = s + it^{1/2-\alpha} \\ s \in [-k_0 t^{1/2-\alpha}, k_0 t^{1/2-\alpha}]}} e^{imyt^{-1/2}} F(m, t)\, dm$$

$$= \frac{1}{t}\left(1 + y^2/t\right)^{-1}\left[\int e^{imyt^{-1/2}}\left(I - \partial_m^2\right) F(m,t)\, dm + B(y,t)\right],$$

where $B(y,t)$ is uniformly bounded for all $y \in \mathbb{R}$ and t bounded away from zero, and

$$F(m,t) = e^{-\tau_2 m^2} \exp\left[\left(\mu(mt^{-1/2}) - \tau_2 m^2 t^{-1}\right) t\right] mQ(mt^{-1/2}).$$

The function $\mu(mt^{-1/2}) - \tau_2 m^2 t^{-1}$ is $\mathcal{O}(t^{-N\alpha})$ for $|m| \leqq Ct^{1/2-\alpha}$. Furthermore, it is analytic in $mt^{-1/2}$, so its derivatives are $\mathcal{O}(t^{-N\alpha})$ as well. Claim 2 follows.

Proof of Claim 3: Claim 3 follows by rewriting the integral in 3.8 as in the proof of Claim 2, and observing that $e^{\mu(z)t} - e^{-\tau_2 z^2 t} = \mathcal{O}(t^{1-N\alpha})$ for $|z| < t^{-\alpha}$.

The only contribution to the integral (3.6) which we have not estimated is

$$I = \int_{\substack{z = r(\pm k_0 + i\eta) \\ r \in [0, t^{-\alpha}]}} e^{izy} e^{-\tau_2 z^2 t}\, dz.$$

But it is simple to show that $I \sim \int_{k \in \mathbb{R}} e^{iky} e^{-\tau_2 k^2 t}$ in L^p $(1 \leqq p \leqq \infty)$ as $t \to \infty$, completing the proof of the Lemma.

4. Nonlinear decay rates

We now return to the full nonlinear problem

$$\begin{pmatrix} U \\ \delta \end{pmatrix}_t = A \begin{pmatrix} U \\ \delta \end{pmatrix} + \begin{pmatrix} \mathrm{NL1}(U,\delta) + \mathrm{NL2}(U,\delta) \\ \beta_y + (\alpha - \alpha_*)_y \end{pmatrix}, \tag{4.1}$$

where A is the linear operator in (1.13), $q = 1/2$,

$$\alpha_*(y,t) = \frac{-1}{\int \varphi'\, dx} \int g''(\varphi(x-\delta(y,t)))\varphi'(x-\delta(y,t))U(x,y,t)\, dx,$$

(4.2)

$$\begin{aligned} \mathrm{NL1}(U,\delta) =& qU_x^2 + \alpha_*(y,t)\varphi'(x)\delta_y \\ &+ \delta_y U \partial_x g'(\varphi(x-\delta)) - \beta\varphi'(x-\delta)\delta_y - W_y + \varphi'(x-\delta)\delta_y^2, \\ & rU_x^2 = \beta(y,t)\varphi'(x-\delta) + W_x, \quad W \in \mathcal{L}^2 \\ & \text{where } g(\varphi+v) = g(\varphi) + g'(\varphi)v + r(\varphi,v)v^2, \end{aligned}$$

and $\mathrm{NL2}(U,\delta)$ represents terms arising from the centering of $\varphi(x-\delta)$ around $\varphi(x)$; that is, if $\delta_*(x,y,t)$ is chosen so that

$$\varphi(x-\delta) = \varphi(x) + \delta\varphi'(x-\delta_*),$$

or

$$g'(\varphi(x-\delta))\varphi'(x-\delta) = g'(\varphi(x))\varphi(x) + \delta\partial_x[g'(\varphi(x-\delta_*))\varphi'(x-\delta_*)],$$

etc., then

(4.3)

$$\begin{aligned} \mathrm{NL2}(U,\delta) =& -\delta U_x \partial_x f'(\varphi(x-\delta_*)) + \delta\delta_y\partial_x g(\varphi(x-\delta_*)) + (\alpha-\alpha_*)\varphi'(x)\delta_y \\ &+ \alpha_*\delta\varphi''(x-\delta_*)\delta_y + (\alpha-\alpha_*)\delta\varphi''(x-\delta_*)\delta_y \\ &- \delta U_y \partial_x g'(\varphi(x-\delta_*)) - (V-V_*)_y. \end{aligned}$$

where

$$V_*(x,y,t) = \begin{cases} -\int_{-\infty}^x g''(\varphi(s))\varphi'(s)U(s)\, ds & (x<0); \\ \int_x^\infty g''(\varphi(s))\varphi'(s)U(s)\, ds & (x>0) \end{cases}$$

(the value of δ_* may be different in each term of (4.3)).

Examining the integral form of (4.1) reveals the type of estimates we will need in order to finally determine long-time behavior in the weighted space L^1_B. Specifically, if $p = 1, 2, \infty$, $\rho = 0, 1$, and ∂ represents ∂_x or ∂_y as applicable, then

(4.4)
$$\begin{aligned}
\|\partial^\rho U(t)\|_{L^p_B} \leqq{}& \|\partial^\rho G_{11}(t)\|_{L^p}\|U(0)\|_{L^1_B} + \|\partial^\rho G_{12}(t)\|_{L^p}\|\delta(0)\|_{L^1} \\
&+ \int_0^t \|\partial^\rho G_{11}(t-s)\|_{L^p}\left(\|\mathrm{NL1}(s)\|_{L^1_B} + \|\mathrm{NL2}(s)\|_{L^1_B}\right)\, ds \\
&+ \int_0^t \|\partial^\rho G_{12}(t-s)\|_{L^p}\left(\|\beta_y(s)\|_{L^1} + \|(\alpha-\alpha_*)_y(s)\|_{L^1}\right)\, ds \\
\|\partial^\rho \delta(t)\|_{L^p_B} \leqq{}& \|\partial^\rho G_{21}(t)\|_{L^p}\|U(0)\|_{L^1_B} + \|\partial^\rho G_{22}(t)\|_{L^p}\|\delta(0)\|_{L^1} \\
&+ \int_0^t \|\partial^\rho G_{21}(t-s)\|_{L^p}\left(\|\mathrm{NL1}(s)\|_{L^1_B} + \|\mathrm{NL2}(s)\|_{L^1_B}\right)\, ds \\
&+ \int_0^t \|\partial^\rho G_{22}(t-s)\|_{L^p}\left(\|\beta_y(s)\|_{L^1} + \|(\alpha-\alpha_*)_y(s)\|_{L^1}\right)\, ds.
\end{aligned}$$

(A minor modification of this scheme will be required in order to estimate $\|\partial_x U\|_{L^2_B}$.)

The necessary estimates on the Green's functions G_{ij}, and on their derivatives, are consequences of Lemma 3.3 when t is bounded away from zero. For t near zero, they can be obtained by a scaling argument, from the invariance of (1.1) under the transformations

$$\begin{aligned}
&t \to \epsilon t, \\
&x \to \sqrt{\epsilon}x, \quad y \to \sqrt{\epsilon}y, \qquad (4.5)\\
&f \to f/\sqrt{\epsilon}, \quad g \to g/\sqrt{\epsilon}.
\end{aligned}$$

Conjugation by the weight $e^{u_l|x|/2}$ does not affect the resulting short-time behavior of the G_{ij} and their derivatives. As $t \to 0$, therefore, we have

$$
\begin{aligned}
&\|\partial^\rho G_{ij}\|_{L^1} = \mathcal{O}(t^{-|\rho|/2}),\\
&\|\partial^\rho G_{ij}\|_{L^2} = \mathcal{O}(t^{1/4-|\rho|/2}) \text{ for } ij \neq 11,\\
&\|\partial^\rho G_{11}\|_{L^2} = \mathcal{O}(t^{1/2-|\rho|/2}),\\
&\|\partial^\rho G_{2j}\|_{L^\infty} = \mathcal{O}(t^{-1/2}).
\end{aligned}
\tag{4.6}
$$

These estimates show that $\|\partial_x G_{11}(t)\|_{L^2}$ is not integrable over a t–interval containing zero. Therefore, as was mentioned earlier, we will not estimate

$$\int_{t-1}^{t} \|\partial_x G_{11}(t-s)\|_{L^2} \left(\|\mathrm{NL1}(s) + \mathrm{NL2}(s)\|_{L^1_B}\right)\, ds.$$

Instead, we will estimate

$$\int_{t-1}^{t} \|G_{11}(t-s)\|_{L^2} \left(\|\partial_x \mathrm{NL1}(s) + \partial_x \mathrm{NL2}(s)\|_{L^1_B}\right)\, ds.$$

In this section we obtain bounds on the L^1 norms (over $\mathbb{R}$ or $\mathbb{R}^2$ as appropriate) of nonlinear terms under the integral signs in (4.4), and of their x-derivatives where applicable. These bounds are obtained in terms of $\|U\|_{L^1_B}$, $\|U_x\|_{L^1_B}$ and $\|U_y\|_{L^1_B}$, as well as $\|U\|_{L^2_B}$, $\|U_x\|_{L^2_B}$, $\|\delta_y\|_{L^2}$, $\|\delta_y\|_{L^1}$, and $\|\delta\|_{L^\infty}$.

Several estimates from [5] are useful at this stage. These include bounds of the forms

$$
\begin{aligned}
&\alpha^2(y,t) \leqq C(u_l; g) \int |U(x,y,t)|\, dx,\\
&|\beta(y,t)| \leqq C \int U_x^2(x,y,t)\, dx,
\end{aligned}
\tag{4.7}
$$

as well as global (in time) bounds on q, r and their derivatives, and on $\|U\|_{L^\infty}$, $\|\delta\|_{L^\infty}$ and $\|\delta_y\|_{L^\infty}$. A global bound on $\|U_x\|_{H^2}$, and thus on $\|U_x\|_{L^\infty}$, can be obtained by energy estimates in the spirit of [5].

The estimates on the nonlinear terms in (4.4) are mostly routine, requiring Schwartz and Sobolev inequalities, as well as the recognition that $|\delta_*(\cdot, y, t)| \leqq |\delta(y,t)|$ and therefore any function ψ with uniformly bounded x–derivative satisfies $|\psi(x-\delta_*)| \leqq C|\psi(x)|$ for some C independent of x, y and t. We present only some of the calculations.

First,

$$\begin{aligned}\|\alpha_*(y,t)\varphi'(x)\delta_y\|_{L^1_B} &\leqq C\|\alpha_*\|_{L^2}\|\delta_y\|_{L^2} \\ &\leqq C\|U\|_{L^2}^2 + C\|\delta_y\|_{L^2}^2 \text{ by (4.7);}\end{aligned}$$

the terms $\beta\varphi'(x-\delta)\delta_y$, $\alpha_*\delta\varphi''(x-\delta_*)\delta_y$ and $\delta_y U\partial_x(g'(\varphi(x-\delta)))$ are estimated similarly.

Next,

$$\begin{aligned}\|\beta_y\|_{L^1} &\leqq C\|\delta_y\|_{L^\infty}\|U_x\|_{L^2}^2 + C(1+\|U_x\|_{L^\infty})\|U_xU_{xy}\|_{L^1} \text{ by (4.7)} \\ &\leqq C\|U_x\|_{L^2}^2 + C\|U_x\|_{L^2}^{3/2}\|U_x\|_{H^2}^{1/2} \\ &\leqq C\|U_x\|_{L^2}^2 + C\|U_x\|_{L^2}^{3/2}\end{aligned}$$

and

$$\|\delta U_x\partial_x f'(\varphi(x-\delta_*))\|_{L^1_B} \leqq C\|\delta\|_{L^\infty}\|U_x\|_{L^1_B};$$

the term $\delta\delta_y\partial_x g(\varphi(x-\delta_*))$ is estimated similarly to $\delta U_x\partial_x f'(\varphi(x-\delta_*))$.

Also,

$$\begin{aligned}\|(\alpha-\alpha_*)\varphi'(x)\delta_y\|_{L^1_B} &\leqq C\|\delta\delta_y\int U\partial_x\left[g''(\varphi)\varphi'(x-\delta)\right]\,dx\|_{L^1} \\ &\leqq C\|\delta\|_{L^\infty}\|\delta_y\|_{L^1};\end{aligned}$$

similar estimates apply to the terms $(\alpha-\alpha_*)\delta_y\delta\varphi''(x-\delta_*)$, $U_y\delta\partial_x g'(\varphi(x-\delta_*))$, and $(\alpha-\alpha_*)_y$.

An integral term in W_y merits attention (a term in $(V_*-V)_y$ is handled similarly). For $x<0$, we have

(4.8)
$$W_y = \int_{-\infty}^{x}\left[2rU_xU_{xy} + \left(-r_\varphi\varphi'\delta_y + r_{U_x}U_{xy}\right)U_x^2\right]\,dx' - \beta_y\left[\varphi(x-\delta)\right] + \delta_y\beta\varphi'.$$

Since $\|U_x\|_{L^2_B}$ is known to be finite, we can integrate $e^{u_l|x|/2}$ times the first term in (4.8) by parts, yielding

$$\begin{aligned}\int_{y=-\infty}^{\infty}\int_{x=\infty}^{0} e^{u_l|x|/2}\int_{-\infty}^{x}\left|2rU_xU_{xy}\right|\,dx'\,dx\,dy &\leqq \\ &\leqq C\int_{y=-\infty}^{\infty}\int_{x=\infty}^{0} e^{u_l|x|/2}\left|2rU_xU_{xy}\right|\,dx\,dy \\ &\leqq C\|U_x\|_{L^2_B}\|U_{xy}\|_{L^2} \leqq C\|U_x\|_{L^2_B}^{3/2}.\end{aligned}$$

To obtain decay rates for U_x and $\delta-\delta_0$ is now straightforward. The procedure requires two steps.

For the first step, define

$$\begin{aligned} a(t) &= \|U(t)\|_{L^2_B} + \|U_x(t)\|_{L^2_B} + \|\delta_y(t)\|_{L^2}, \\ b(t) &= \|U_x(t)\|_{L^1_B} + \|U_y(t)\|_{L^1_B} + \|\delta_y(t)\|_{L^1} + \|\delta(t)\|_{L^\infty}. \end{aligned} \tag{4.9}$$

As a result of the estimates sketched above, all the norms of the nonlinear terms under the integral sign in (4.4) can be bounded by a constant times $\left(a^2+a^{3/2}+b^2\right)(s)$. Applying the bounds on the Green's functions from (4.6) and Lemma 3.3 therefore yields

$$\begin{aligned} a(t) &\leqq Ct^{-3/4} + CI_0\int_0^t (t-s)^{-3/4}\left(a^2+a^{3/2}+b^2\right)(s)\,ds, \\ b(t) &\leqq Ct^{-1/2} + CI_0\int_0^t (t-s)^{-1/2}\left(a^2+a^{3/2}+b^2\right)(s)\,ds, \end{aligned}$$

where

$$I_0 = a(0) + b(0) + \|\delta(0)\|_{L^1} + \|U(0)\|_{L^1_B}.$$

If $T > 0$ and $M(T) = \sup_{[0,T]} \left(t^{3/4}a + t^{1/2}b\right)$, it follows that

$$\begin{aligned} M \leqq CI_0 &+ C^{1/2}\int_1^t (t-s)^{-1/2}s^{-1}(M^2+M^{3/2})\,ds \\ &+ Ct^{1/2}\int_0^1 (t-s)^{-1/2}\left(a^2+a^{3/2}+b^2\right)(s)\,ds \\ &+ Ct^{3/4}\int_1^t (t-s)^{-3/4}s^{-1}(M^2+M^{3/2})\,ds \\ &+ Ct^{3/4}\int_0^1 (t-s)^{-3/4}\left(a^2+a^{3/2}+b^2\right)(s)\,ds. \end{aligned}$$

Therefore $M \leqq C\left(I_0 + M^2 + M^{3/2}\right)$, and so M is bounded for all T if the initial values I_0 are small enough. As $t \to \infty$, therefore, we have

$$a(t) = \mathcal{O}(t^{-3/4}),$$

$$b(t) = \mathcal{O}(t^{-1/2}).$$

The second and final step is to bring the leading-order term $G_{22} * \delta(0)$ to the left–hand side of (3.3). Estimates like those above then show that

$$\|\delta(t) - G_{22}(t) * \delta(0)\|_{L^1} = \mathcal{O}(t^{-1/2})$$

as $t \to \infty$. This completes the proof of Theorem 1.2.

Bibliography

[1] Bronsard, L. and Hilhorst, D., *On the slow dynamics for the Cahn-Hilliard equation in one space dimension*, Proc. Roy. Soc. London Ser. A 439 (1992), 669–682.

[2] Caginalp, G., *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math. 44 (1990), 77–94.

[3] Gelfand, I.M., *Some problems in the theory of quasilinear equations*, Uspekhi Mat. Nauk (N.S.) 14 (1959), 87–158; English transl., Amer. Math. Soc. Transl. (2) 29 (1963).

[4] Goodman, J., *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rational Mech. Anal. 95 (1986), 325–344.

[5] Goodman, J., *Stability of viscous scalar shock fronts in several dimensions*, Trans. AMS 311 (1989), 683–695.

[6] Kato, T., Perturbation theory for linear operators, New York, Springer, 1976.

[7] Kawashima, S., and Matsumura, A., *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys. 101 (1985), 97–127.

[8] Jones, C.K.R.T., Gardner, R., and Kapitula, T., *Stability of travelling waves for nonconvex scalar viscous conservation laws*, Comm. Pure Appl. Math. 46 (1993), 505–526.

[9] Liu, T.P., *Nonlinear stability of shock waves for viscous conservation laws*, Mem. Amer. Math. Soc., vol. 328, 1985.

[10] Pego, R.L., *Front migration in the nonlinear Cahn-Hilliard equation*, Proc. Roy. Soc. London Ser. A 422 (1989), 261–278.

[11] Sattinger, D., *On the stability of waves of nonlinear parabolic systems*, Adv. in Math 22 (1976), 312–355.

[12] Szepessy, A., and Xin, Z.P., *Nonlinear stability of viscous shock waves*, Arch. Rational Mech. Anal. 122 (1993), 53–103.

[13] Taylor, G.I., *Dispersion of soluble matter in solvent flowing slowly through a tube*, Proc. Roy. Soc. London A 219 (1953), 186–203.

[14] Van den Broeck, C., *Taylor dispersion revisited*, Phys. A 168 (1990), 677–696.

[15] Zeng, Y., *L^1 asymptotic behavior of compressible, isentropic, viscous 1-D flow*, Comm. Pure Appl. Math. 47 (1994), no. 8, 1053–1082.