Final Exam, must be uploaded to the class NYU Classes site by 11:55 pm (five minutes before midnight), Tuesday, May 19, New York City time

Corrections: May 13. Question 7 fixed to make $\chi_\rho(\text{id})$ an integer multiple of $|G|$. May 14. Fixed a typo of question 4 to make it $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$ instead of the incorrect $\mathbb{Q}$ and $\mathbb{Z}$. Again, May 14, question 4 simplified to be correct. May 16, Question (3j) modified to say irreducible "over" $\mathbb{Q}[\xi]$ instead of "in" and to make clear the distinction.

Instructions, please read carefully.

- Email me immediately if you have a question or suspect there is a problem with a question. If there are corrections, I will communicate these through the NYU Class announcements system. Make sure you get such announcements.
- Do not post questions about or comments on this exam on the class forum or anywhere else. It is an exam.
- Do this exam independently. Do not communicate with anyone but me about it before the submission deadline (midnight Tuesday, May 19) has passed. Do not consult any web resources. You may refer to class notes and the four class sources (the E. Artin book, the M. Artin book, and the two J.P. Serre books).
- You will be graded on the correctness of writing and exposition as well as mathematical correctness. Please take the time to explain your answers in complete correct sentences using correct terminology. This includes using the correct logical works (e.g., not "thus" for "in addition to"). That said, there may be different ways to answer a given question and different ways to explain a given answer.
- You will receive 20% credit for a blank answer. Points will be deducted for anything wrong, even if the right answer also appears.
- If you cannot answer one part of a multi-part question, you may leave that blank and assume the answer in other parts.
- If you type answers, please use generous spacing that makes your answers easy to read. If you write your answers by hand (as I would), please leave generous spacing and make sure that what you upload is easy to read.

Questions.

1. Let V and W be vector spaces over a field $\mathbb{K}$, and let $A\colon V \to W$ a linear map. Do not assume V or W are finite dimensional. Suppose $x_k \in V$ for $k = 1, \ldots, n$. Show that if the family $y_k = Ax_k$ is linearly independent in W, then the family $x_k \in V$ is linearly independent.

2. Let $\mathbb{E} = \mathbb{K}(t)$ be the field of rational functions over $\mathbb{K}$. An element $u \in \mathbb{E}$ may be expressed as a ratio of polynomials
$$u(t) = \frac{f(t)}{g(t)} \,,$$
or as a sum of other rational functions, such as $u(t) = 3t - 2(t-1)^{-2}$. Let $\sigma\colon \mathbb{E} \to \mathbb{E}$ be the mapping
$$u(t) \xrightarrow{\sigma} u\left(t^{-1}\right) \,.$$
This is the same as saying that if $v = \sigma(u)$, then $v(t) = u(t^{-1})$. Let $\mathbb{F} = \mathbb{E}^{\sigma}$ be the fixed field of σ.
   (a) Show that σ is an automorphism of $\mathbb{E}$.
   (b) Show that $\mathbb{E}$ is a finite degree extension of $\mathbb{F}$, and find the degree.
   (c) Find a polynomial
   $$p(u) = u^n + p_{n-1}u^{n-1} + \cdots + p_0$$
   with coefficients $p_k \in \mathbb{F}$ so that $\mathbb{E}$ is the splitting field of p over $\mathbb{F}$. *Hint*: You know $u \notin \mathbb{F}$ if $\sigma(u) = -u$.

3. Let $\mathbb{E} \subset \mathbb{C}$ be the splitting field over $\mathbb{Q}$ of the polynomial $f(x) = x^6 + 2$.
   (a) Show that $\mathbb{E}$ contains a primitive sixth root of unity, ξ.
   (b) Let $\mathbb{F} = \mathbb{Q}[\xi]$. Show that $\mathbb{F}$ contains all sixth roots of unity.
   (c) Define a polynomial
   $$g(x) = \prod_{j=0}^{5} \left(x - \xi^j\right) \,.$$
   Show that $g(x) = x^6 - 1$.
   (d) Find the factorization of g into irreducible monic polynomials in $\mathbb{Q}[x]$. Say which roots of unity ξ^j are roots of each factor. *Hint*: One of the factors involves the primitive cube roots of unity.
   (e) Show that there are six distinct numbers $\alpha_j \in \mathbb{C}$ with $\alpha_j^6 = -2$.
   (f) Show that $f(x) = x^6 + 2$ is irreducible in $\mathbb{Q}[x]$. *Hint*: If $f = f_1(x)f_2(x)$ then in $\mathbb{E}$, some of the α_j are roots of f_1 and others are roots of f_2 (and similarly if there are more factors). If α_1 is a root of f_1, then the field $\mathbb{Q}[\alpha_1]$ also contains $-\alpha_1$, so the splitting field of f_1 over $\mathbb{Q}$ contains $-\alpha_1$, so $-\alpha_1$ is a root of f_1. This limits the degree and form of factors f_1 and f_2.

(g) Show that the field $\mathbb{K} = \mathbb{Q}[\alpha]$ (adjoining a single root of f) does not contain a primitive sixth root of unity.

(h) Show that $\mathbb{K} = \mathbb{Q}[\alpha]$ is a degree 6 extension of $\mathbb{Q}$ that is not a Galois (normal) extension.

(i) Show that $\mathbb{E}$ is a degree 2 extension of $\mathbb{K}$ and a degree 12 extension of $\mathbb{Q}$.

(j) Show that f is irreducible over $\mathbb{F} = \mathbb{Q}[\xi]$. This means that f is irreducible in $\mathbb{F}[x]$.

4. For this question M is a module over $\mathbb{Z}$ with $M \subset \mathbb{Q}$ and $\mathbb{Z} \subset M$, all operations are addition or multiplication in $\mathbb{Q}$. Give an example of such an M that is finitely generated and an example of such an M that is not finitely generated.

5. This is an exercise involving finite fields and quadratic reciprocity.

   (a) Show that 127 is a prime number. This should not involve more than a few calculations. Do not make a sieve up to 127.

   (b) Show that $f(x) = x^2 - 3$ is irreducible in $\mathbb{F}_{127}$. Use Galois theory and quadratic reciprocity.

   (c) Let $\mathbb{E}$ be the splitting field of f over $\mathbb{F}_{127}$. Show that if $\alpha^2 = 3$ in $\mathbb{E}$, and $\beta = \alpha^{127}$, then $\beta^2 = 3$. Your argument should apply with 127 and 3 replaced by any prime and number that is not a quadratic residue for that prime.

6. Let G be a finite group that is an extension of another finite group H. This means that H is a subgroup of G. Let V be a finite dimensional vector space over $\mathbb{C}$. Let σ be a representation of G on V and ρ a representation of H. Then σ is an extension of ρ if $\sigma(h) = \rho(h)$ for all $h \in H$. Consider the following conjecture: Given any finite dimensional representation ρ of a finite group H on V, there is an extension group G and an extension representation σ so that V is irreducible over σ. Either prove the conjecture true or prove that it is not true (possibly by giving a counter-example).

7. Let $\chi_\rho(g)$ be the character of a finite dimensional representation of a finite group. Suppose that $\chi_\rho(g) = 0$ if $g \neq \text{id}$. Show that $\chi_\rho(\text{id}) = m\,|G|$ for some integer m, and that ρ is isomorphic to a direct sum of m copies of the regular representation. Here, $|G|$ is the order of G.