Assignment 10, due April 15

Corrections: Due date moved to Wednesday, Exercise 4 removed.

1. Let C_n be the cyclic group of order n with elements $\{0, 1, \ldots, n-1\}$. The group operation is addition mod n. Show that for each $\alpha \in \{0, \ldots, n-1\}$ there is a two dimensional representation over $\mathbb{R}$ given by rigid rotations in the plane:

$$k \to \rho_\alpha(k) = \begin{pmatrix} \cos\left(\frac{2\pi\alpha k}{n}\right) & \sin\left(\frac{2\pi\alpha k}{n}\right) \\ -\sin\left(\frac{2\pi\alpha k}{n}\right) & \cos\left(\frac{2\pi\alpha k}{n}\right) \end{pmatrix} .$$

   (a) Verify that the ρ_α are linear representations of C_n over $\mathbb{R}$ and over $\mathbb{C}$.

   (b) Show that ρ_α is irreducible over $\mathbb{R}$ if $\alpha \neq 0$.

   (c) Determine the relation between α and β that is equivalent to ρ_α being isomorphic to ρ_β.

   (d) Express the complex representation space as a direct sum of subspaces invariant under ρ_α. That is, find one dimensional subspaces $W^+ \subset \mathbb{C}^2$ and $W^- \subset \mathbb{C}^2$ that are invariant under the action of ρ_α. Show that the characters of these irreducible representations are $\chi^{\pm}_{\rho_\alpha}(k) = e^{\pm\frac{2\pi i\alpha k}{n}}$

2. Construct a 4 dimensional representation of C_8 over $\mathbb{Q}$ as follows. Define the 2×2 rational matrix

$$a = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} .$$

   Define the 4×4 rational matrix as a 2×2 *block matrix* (a matrix whose entries are matrices)

$$A = \begin{pmatrix} a & a \\ -a & a \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 \\ \hline 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 \end{pmatrix} .$$

   (a) Show that $A^k = I$ if and only if $k \equiv 0 \bmod 8$. *Hint*: First calculate a^2. Then use the 2×2 block matrix form of A, not the 4×4 element form, to calculate A^2 in block matrix form. Compute A^4 as the square of A^2. You will see what all the powers of A are.

(b) Show that $\rho_{\mathbb{Q}}(k) = A^k$ is a representation of C_8 over $\mathbb{Q}$ in $V = \mathbb{Q}^4$.

(c) Show that $\rho_{\mathbb{Q}}$ is irreducible over $\mathbb{Q}$. *Hint*: no 1×1 or 2×2 or 3×3 rational matrix can have (complex) eigenvalues consistent with part (a).

(d) (*extra credit*) Consider $\rho_{\mathbb{Q}}$ as a real representation in $V = \mathbb{R}^4$. Show that $V = W^1 \oplus W^2$, where $\rho_{\mathbb{Q}}$ over $\mathbb{R}$, where W^1 and W^2 are representations of the kind introduced in exercise 1, with $n = 8$.

[*Explanation*: The representation $\rho_{\mathbb{Q}}$ was constructed with part (d) in mind. With $n = 8$, exercise 1 uses the rotation matrix by angle $\frac{\pi}{4}$, which is

$$b = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} .$$

The formula for A has this form with a instead of $\frac{1}{\sqrt{2}}$. The matrix a is a 2×2 rational matrix that "acts like" $\pm\frac{1}{\sqrt{2}}$ in that the eigenvalues are $\pm\frac{1}{\sqrt{2}}$.]

3. In the notation of *Linear Representations of Finite Groups*, page 20. Let v_1 and v_2 be the basis for the 2D representation space for the representation of S_3 with character θ:

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} .$$

The representation space is $V \subset \mathbb{C}^3$ with $x_1 + x_2 + x_3 = 0$. A permutation $\pi \in S_3$ acts on $x \in \mathbb{C}^3$ by permuting the components:

$$\rho_\pi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_{\pi(1)} \\ x_{\pi(2)} \\ x_{\pi(3)} \end{pmatrix} .$$

The notation is $(1, 2, 3) \xrightarrow{\pi} (\pi(1), \pi(2), \pi(3))$. Let C be the cyclic permutation $(1, 2, 3) \xrightarrow{C} (2, 3, 1)$.

(a) Find the 2×2 matrix r_C that represents ρ_C in the v_1, v_2 basis. Verify directly that $\mathrm{Tr}(r_c) = -1$.

(b) Find the one dimensional subspaces $W \subset V$ that are invariant under the cyclic subgroup of S_3 generated by C. Show that this is the same as finding vectors $z \in V$ so that $r_C z = \alpha z$. Find two such vectors, linearly independent.

(c) Verify directly without using character theory that this two dimensional representation is irreducible. Do this by showing that there is no one dimensional invariant subspace. That means, there is no $y \in V$ with $r_C y = \alpha y$ and $r_T y = \beta y$. Here, r_T is the 2×2 matrix corresponding to the transposition $(1,2,3) \xrightarrow{T} (2,1,3)$. You can do this directly, or showing that the subspaces W from part (b) are not invariant under T.

4. (*Removed*)

5. (*Preparation for exercise 6*) Let $f(y)$ be a twice differentiable real function of n real variables $y_1, \ldots, y_n$. Let Q by an $n \times n$ matrix and define $g(x) = f(Qx)$.

   (a) Find a formula for $\partial_{x_j} g$ in terms of the derivatives $\partial_{y_k} f$ and the entries of Q.

   (b) Let $a = (a_1, \ldots, x_n)^t$ be an $n-$component real column vector. Define row vectors $\nabla f = (\partial_{y_1} f, \ldots, \partial_{y_n} f)$. The *directional derivative* of f in the direction a is the matrix product of the row vector and column vector, written in various ways

$$\nabla f \cdot a = \sum_{j=1}^{n} a_j \partial_{y_j} f = (a \cdot \nabla) f = a \cdot \operatorname{grad} f \ .$$

   Let $\nabla g = (\partial_{x_1} g, \ldots, \partial_{x_n} g)$. Show that $\nabla g \cdot b = \nabla f \cdot a$ for some column vector b and find a matrix/vector formula involving Q that relates b and a. *Hint*: One way is to use ordinary partial derivatives and the chain rule, then interpret the result in terms of matrix and vector operations.

   (c) The matrix of second partial derivatives of f, often called the *Hessian* matrix, is

$$D^2 f = \begin{pmatrix} \partial_{y_1}^2 f & \partial_{y_1}\partial_{y_2} f & \cdots & \partial_{y_1}\partial_{y_n} f \\ \partial_{y_1}\partial_{y_2} f & \partial_{y_1}^2 f & & \partial_{y_1}\partial_{y_n} f \\ \vdots & & \ddots & \vdots \\ \partial_{y_1}\partial_{y_n} f & & & \partial_{y_n}^2 f \end{pmatrix}$$

   Suppose R is a symmetric $n \times n$ matrix. Define (in various notations, note $R_{jk} = R_{kj}$)

$$R :: D^2 f = \operatorname{Tr}(R D^2 f) = \sum_{j=1}^{n} \sum_{k=1}^{n} R_{jk}\, \partial_{y_j} \partial_{y_k} f \ .$$

   Show that $R :: D^2 f = S :: D^2 g$ and find a formula involving Q that relates R to S.

6. Group representations are used in physics and chemistry to understand functions or motions that respect certain groups of symmetries. A *point group* of symmetries is a group that represents *rigid rotations* about a fixed point. Mathematically, a rigid rotation is represented by an *orthogonal*

matrix, Q. A real $n \times n$ matrix Q is *orthogonal* if $QQ^t = I$. Geometrically, Q being orthogonal means that it does not change the length of vectors or the angles between vectors. If $y = Ax$, then $\|y\|_2 = \|x\|_2$. If $v = Qu$, then $\langle v, y\rangle = v^t y = \langle u^t x\rangle = u^t x$.

(a) Check that the set of orthogonal $n \times n$ matrices forms a group. This is called the *orthogonal group* and written O_n, or $\mathrm{O}(n)$, or $\mathrm{O}(n, \mathbb{R})$. Show that if $Q \in \mathrm{O}_n$, then $\det(Q) = \pm 1$. The subset of O_n with $\det(Q) = 1$ is the *special* orthogonal group, written SO_n.

(b) Let F_d be the set of homogeneous polynomials of degree d in n variables ("F" is because homogeneous polynomials are often called "forms"). Show that O_n acts on F_d and that this is a linear representation of O_n. Let $\mathrm{GL}(F_d)$ be the group of linear transformations on F_d. For $Q \in \mathrm{O}_n$, define $\rho(Q) \in \mathrm{GL}(F_d)$ as follows. If $f \in F_d$, then $g = \rho(Q)f$ is defined by $g(x) = f(Qx)$.

(c) An *operator* is a linear map: function $\rightarrow$ function. The *Laplace operator* (or *laplacian*) $\triangle$ is defined by

$$\triangle f(x) = I :: D^2 f = \mathrm{Tr}(D^2 f) = \sum_{j=1}^{n} \partial_{x_j}^2 f(x) \; .$$

For example, in 2D,

$$\triangle f(x, y) = \left(\partial_x^2 + \partial_y^2\right) f(x, y) \; .$$

A function f is *harmonic* if $\triangle f = 0$. Let H_d be the set of $f \in F_d$ that are harmonic. Show that O_n acts on H_d as a sub-representation. *Hint*: This is a fancy way to say that if $f(x)$ is harmonic, then $f(Qx)$ is harmonic. You can start by exploring the example $n = 2$ and $d = 2$, and $f(x, y) = x^2 - y^2$. A $Q \in \mathrm{SO}_2$ has the form

$$Q = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \; .$$

Calculate $g(x, y) = \rho(Q)(x^2 - y^2)$ and check directly that it is harmonic.

(d) Find the dimension of and a basis for H_d in two dimensions. Show that you may take as a basis $\mathrm{Re}\left((x + iy)^n\right)$ and $\mathrm{Im}\left((x + iy)^n\right)$. *Hint*: There may be an elegant way to do this, but a "hands on" or "direct" method is to say if f has x^n then it must have $x^{n-2}y^2$ with a certain coefficient, and so on.

(e) Find the dimension and a basis for H_d when $n = 3$ for $d = 0, 1, 2$. Show that $F_2 = H_2 \oplus r^2 H_0$, where $r^2 = (x^2 + y^2 + z^2)$.

(f) (*extra credit*) Find the dimension and a basis for H_3 for $d = 3$. Show that $F_3 = H_3 \oplus r^2 H_1$. Before you start, be aware that the dimensions are $10 = 7 + 3$. See the comments below for more information.

Comments: A point on the unit sphere will be called $\omega = \frac{x}{\|x\|_2}$. Any $f \in F_d$ may be written $f(x) = \widetilde{f}(\omega) r^n$, where $r = \|x\|_2$. If f is harmonic, the corresponding $\widetilde{f}$ is a *spherical harmonic*. In 3D, $\dim(H_d) = 2d + 1$, which is 3 (with basis x, y, z) for linear polynomials, $d = 5$ for quadratic polynomials, and so on. In any dimension there is a direct sum representation $F_n = H_n \oplus r^2 H_{n-2}$. That means that for f a homogeneous polynomial of degree n, there is a harmonic polynomial h of degree n and a homogeneous polynomial g of degree $n - 2$ so that $f(x) = h(x) + \|x\|_2^2 \, g(x)$. Note that $\|x\|_2^2 \, g(x) = (x_1^2 + \cdots + x_n^2) g(x)$ is a homogenous polynomial of degree n. If you like combinatorics, you can see that

$$\dim(F_d) = \binom{n+d}{n} .$$

For dimension $n = 3$, (if you like doing algebra) this confirms that $\dim(H_d) = \dim(F_d) - \dim(F_{d-2}) = 2d + 1$. The spaces of spherical harmonics, H_d, turn out to be the irreducible representations of SO_n. The representation F_n has decomposition into irreducible representations $F_d = H_d \oplus H_{d-2} \oplus \cdots$.

7. (*extra credit*) Let R be a ring without zero divisors (an *integral domain*, or just *domain*). Let K be the field of fractions. Let $M \subset K$ be a module over R. Such a module is a *fractional ideal* if there is an $a \in R$ so that $aM \subseteq R$. The ring R is considered a fractional ideal. If M_1 and M_2 are fractional ideals, their *product* is the set of finite sums from M and N. The number of terms (the range of j in the sum below) is arbitrary, but finite.

$$M_1 \cdot M_2 = \left\{ \sum_j x_j y_j \;\middle|\; x_j \in M_1 \;,\;\; y_j \in M_2 \right\} .$$

(a) Show that if $M \subseteq R$ and M is a fractional ideal, then M is an ideal. The "improper" ideal $M = R$ is considered an ideal for this purpose.

(b) Show that the the product of fractional ideals is a fractional ideal.

(c) Show that if $M_1 \subseteq R$ and $M_2 \subseteq R$, then $M_1 \cdot M_2 \subseteq R$. The product of ordinary ideals is an ordinary ideal.

(d) Show that fractional ideal multiplication is associative:

$$(M_1 \cdot M_2) \cdot M_3 = M_1 \cdot (M_2 \cdot M_3) \; .$$

Show $M = R$ is the identity element for ideal multiplication. Do not show that fractional ideals form a group (for fractional ideal M_1 there is another fractional ideal M_2 with $M_1 \cdot M_2 = R$). That's harder and isn't always true in this generality. It requires more hypotheses.

(e) For any $c \in K$, show that the principal fractional ideal $(c) = \{ca | a \in R\}$ is a fractional ideal and that multiplication of principal fractional ideals corresponds to ordinary multiplication in R: $(c) \cdot (d) = (cd)$. Show that $(c)M = \{cx \mid x \in M\}$.

(f) For $R = \mathbb{Z}$ and $K = \mathbb{Q}$, show that there is a natural $1-1$ correspondence between fractional ideals and fractions (rational numbers) $c \in \mathbb{Q}$. Every fractional ideal is principal.

(g) Give an example of a submodule $M \subset \mathbb{Q}$ that is not a fractional ideal over $\mathbb{Z}$.