Assignment 13, due May 6

Corrections: none yet.

1. This exercise describes the two dimensional representation of S_3. Take K to be a general field, with extra hypotheses applied as necessary. The trivial representation is the one dimensional representation with $\rho_t(\pi) = 1$ for all $\pi \in S_3$. The alternating representation is the one dimensional representation with $\rho_a(\pi) = \mathrm{sign}(\pi)$. The corresponding characters are $\chi_t(\pi)$ and $\chi_a(\pi)$. For one dimensional representations, there is little difference between the representation and the character. The two dimensional representation is ρ and its character is χ (no subscript).

   (a) Let S_3 act on K^3 by
   $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} x_{\pi(1)} \\ x_{\pi(2)} \\ x_{\pi(3)} \end{pmatrix}$$
   Let $V \subset K^3$ be defined by $x_1 + x_2 + x_3 = 0$. Verify that the following is a basis for V:
   $$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \;,\;\; v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \;.$$

   (b) The permutation (12) is the $2-$cycle $1 \to 2 \to 1$, and $3 \to 3$. That is $\pi(1) = 2$, $\pi(2) = 1$, and $\pi(3) = 3$. Suppose $x = \xi_1 v_1 + \xi_2 v_2$ and $x \xrightarrow{(12)} y = \eta_1 v_1 + \eta_2 v_2$. Write an expression for the matrix $R_{(21)}$ that expresses this operation in the v_1, v_2 basis.
   $$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = R_{(12)} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \;.$$

   (c) Use this expression for $R_{(12)}$ to show that this two dimensional representation has
   $$\chi(\text{two cycle}) = 0 \;.$$
   Explain why knowing $\chi(\,(12)\,)$ tells you $\chi(\text{any two cycle})$.

   (d) Use the method of parts (b) and (c) to calculate $R_{(123)}$ and show that
   $$\chi(\text{three cycle}) = -1 \;.$$

(e) Specialize to $K = \mathbb{C}$. Show that the three representations are orthonormal

$$\begin{aligned}
\langle \chi_t, \chi_t \rangle &= 1 \\
\langle \chi_a, \chi_a \rangle &= 1 \\
\langle \chi, \chi \rangle &= 1 \\
\langle \chi_t, \chi_a \rangle &= 0 \\
\langle \chi_t, \chi \rangle &= 0 \\
\langle \chi_a, \chi \rangle &= 0 \; .
\end{aligned}$$

Explain how this shows the representation (ρ, V) is irreducible.

(f) We saw the formula $|G| = n_1^2 + \cdots$, where n_j are the dimensions of the irreducible representations of G. Use this to show that ρ_t, ρ_a, and ρ are all the irreducible representations of S_3.

(g) Describe the conjugacy classes does S_3 and explain how the number of them gives a different proof that our list of irreducible representations is complete.

(h) Let $\zeta = e^{2\pi i/3}$ be a primitive third root of unity in $\mathbb{C}$. Show that the following vectors are a basis of V

$$w_1 = \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \end{pmatrix} \;, \quad w_2 = \begin{pmatrix} 1 \\ \zeta^2 \\ \zeta \end{pmatrix} \; .$$

Find the matrices $R_{(12)}$ and $R_{(123)}$ in this basis and check that they give the same character.

2. Consider the dihedral group with $n = 2m$ (even). Note that the dihedral group has $|D_n| = 2n = 4m$ elements. Let $\zeta_j = e^{2\pi j/n}$ be an $n-th$ root of unity, with ζ_1 being a primitive root.

(a) Show that D_n is generated by two permutations of $(0, 1, \ldots, n-1)$.

$$\begin{aligned}
(0, 1, \ldots, n-1) &\xrightarrow{\pi_r} (1, 2, \ldots, n-1, 0) \qquad \text{(circular rotation)} \\
(0, 1, \ldots, n-1) &\xrightarrow{\pi_f} (0, n-1, n-2, \ldots, 1) \quad \text{(flip)} \; .
\end{aligned}$$

Show that π_r and π_f satisfy the relations $\pi_r^n = 1$, etc., that define D_n.

(b) Consider the point set $H \subset \mathbb{C}$ ("H" is for "hedron", from "polyhedron") consisting of ζ_j for $j = 0, \ldots, n-1$. Show that the map $z \to \zeta_1 z$ (with $z \in \mathbb{C}$) applies π_r to H and $z \to \overline{z}$ applies π_f. This explains (again) the terminology "rotation" and "flip".

(c) Consider the vectors in $\mathbb{C}^n$

$$v_1 = \begin{pmatrix} 1 \\ \zeta_j \\ \zeta_j^2 \\ \zeta_j^{n-1} \end{pmatrix} \;, \quad v_2 = \pi_f v_1 = \begin{pmatrix} \overline{1} \\ \overline{\zeta_j} \\ \overline{\zeta_j^2} \\ \overline{\zeta_j^{n-1}} \end{pmatrix} \;.$$

Consider the two dimensional space $V = \mathrm{span}(v_1, v_2)$. Show that V is a two dimensional representation of D_n find the matrices $R_j(r)$ and $R_j(r)$. More generally, find $R_j(r^k)$ and $R_j(fr^k)$. Note that these are slightly different from the two dimensional representation matrices given in Serre.

(d) Use these to calculate $\chi(r^k)$ and $\chi(fr^k)$. The results should agree with the characters given in Serre.

(e) For the j values that correspond to irreducible two dimensional representations, find two distinct two dimensional subspaces of $\mathbb{C}[D_n]$ that are stable subspaces on which the regular representation is isomorphic the the two dimensional representations above. A two dimensional subspace of $\mathbb{C}[D_n]$ is spanned by two vectors

$$w_1 \in \mathbb{C}^{2n} \;, \quad w_2 \in \mathbb{C}^{2n} \;.$$

These vectors are related to the vectors v_1 and v_2 above. A vector $w \in \mathbb{C}[D_n]$ can be written

$$w = \sum_{k=0}^{n-1} a_k r^k + \sum_{k=0}^{n=1} b_k f r^k \;.$$

Or, you may find another representation more convenient.

3. This set of exercises is related to the Dirichlet theorem that if q is a prime and $a \neq 0 \bmod q$ then there are infinitely many primes $p = a \bmod q$. The exercises are some technical aspects of the proof, but not the whole proof. Half of it is this week and half is in assignment 14. Review the class for the big picture. Some of the exercises involve "analysis", mostly (but not all) at the level of calc 3. If you haven't had the relevant analysis, ignore the exercise. In the terminology from class Wednesday, take a prime q, a generator g of $\mathbb{F}_q^*$. A Dirichlet character is a function of $n \in \mathbb{Z}$ with coset $\overline{n} \in \mathbb{F}_q = \mathbb{Z}/(q)$. The formula is

$$\chi_j(n) = \begin{cases} e^{2\pi i jk/(q-1)} & \text{if } \overline{n} = g^k \text{ in } \mathbb{F}_q^* \\ 0 & \text{if } n = 0 \bmod q. \end{cases}$$

The $j = 0$ character is the *trivial* character. The $q-2$ remaining characters are the *non-trivial* ones. The Dirichlet L function for a specific Dirichlet character is given by the *Dirichlet series*

$$L_j(s) = \sum_{n=1}^{\infty} \frac{\chi_j(n)}{n^s} \;.$$

The Euler product formula for it is (the product is over all primes)

$$L_j(s) = \prod_p \frac{1}{1 - \frac{\chi_j(p)}{p^s}} \ .$$

(a) Show that the Dirichlet character is *multiplicative*, which means that $\chi_j(mn) = \chi_j(m)\chi_j(n)$.

(b) Show that for $s > 1$,

$$\sum_{n=1}^{R} \left| \frac{\chi_j(n)}{n^s} \right| < 1 + \int_1^R x^{-s}\, dx < \frac{1}{s-1} \ .$$

Conclude that the Dirichlet series converges absolutely for $s > 1$.

(c) Show that $L_0(s) \to \infty$ as $s \downarrow 1$. *Hint*: It suffices to consider the terms $n = mq + 1$. Compare this series to an integral as in part (b), but with the integral below instead of above.

(d) Show that the non-trivial characters satisfy

$$\sum_{k=1}^{q} \chi_j(k) = 0 \ .$$

(e) Show that $L_j(s)$ is bounded as $s \downarrow 1$ for the non-trivial characters. That is, show there is an A so that $|L_j(s)| \leq A$ for all $s > 1$. *Hint*: Break the Dirichlet series into a sum over $mq + k$ for $m \geq 0$ and $1 \leq k \leq q$. For the k sum with m fixes, show that

$$\sum_{k=0}^{q-1} \frac{\chi_j(mq+k)}{(mq+k)^s} = \sum_{k=0}^{q-1} \frac{\chi_j(k)}{(mq)^s} + O\left(\frac{1}{m^{s+1}}\right) \ .$$

Then use the result of part (d). The non-trivial Dirichlet series have *cancellation* (nearly equal positive and negative terms cancelling each other in a sum) in the sum that makes the sum bounded even when it stops converging absolutely.

(f) For $s > 1$ consider the functions ϕ_j and ψ_j defined by

$$\phi_j(s) = \log(L_j(s)) = \sum_p \frac{\chi_j(p)}{p^s} + \psi_j(s) \ .$$

Show that $\psi_j(s)$ is bounded as $s \downarrow 1$ for any character (trivial or not). Hint:

$$\log\left(\frac{1}{1 - \frac{\chi_j(p)}{p^s}}\right) = \frac{\chi_j(p)}{p^s} + r \ , \ \ |r| \leq \frac{C}{p^{2s}} \ , \ \ 2s > 2 \ .$$

(g) (*coming next week*)