# Assignment 14, due May 13

**Corrections:** fixed due date. May 12: fixed (3b) to say $s \downarrow 1$ instead of $s \downarrow 0$.

1. Let $G = S_3$. Take $V = \mathbb{C}^2$ spanned by $e_0$ and $e_1$. Define a representation $\rho$ on $V$ as follows. For $\pi \in S_3$, define $\pi e_0 = e_0$ and $\pi e_1 = e_1$ if $\pi$ is an even permutation, and $\pi e_0 = e_1$ and $\pi e_1 = e_0$ if $\pi$ is an odd permutation.
   (a) Show that $\rho$ is irreducible or decompose $\rho$ into its irreducible subrepresentations.
   (b) Show that $\rho$ is the representation induced from the subgroup $C_3 \subset S_3$ from the trivial representation of $S_3/C_3 = C_2$.

2. This sequence of exercises repeats the Lie algebra calculation we did in class for SO(3), but for SU(2). This is the group of $2 \times 2$ complex matrices that are unitary $g^* g = I$ and have determinant $\det(g) = 1$. The Lie algebra of the two groups is the same (isomorphic) and the groups are nearly isomorphic, but not exactly (another interesting but long story).
   (a) A $2 \times 2$ complex matrix $L$ is in the Lie algebra if there is a smooth "path" $g(t) \in \mathrm{SU}(2)$ with $g(0) = I$ and
   $$L = \frac{d}{dt} g(t)\Big|_{t=0} .$$
   Show that if $L$ is in the Lie algebra, then $L$ is skew hermitian and trace free
   $$L^* = -L \;, \;\; \mathrm{Tr}(L) = 0 \; .$$
   Show that this is the same as having real numbers $a$, $b$, and $c$ with
   $$L = \begin{pmatrix} ia & b + ic \\ -b + ic & -ia \end{pmatrix} \; . \tag{1}$$
   The Lie algebra of SU(2) is written $\mathfrak{su}(2)$, which may be made in LaTeX using: "`$\frak{su}(2)$`", but I don't know how to do this in handwriting without making it hard to distinguish from SU(2).
   (b) Define the matrix exponential to be
   $$e^{tL} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n \; .$$
   Show that if $L$ is skew hermitian and trace free, then $g \in \mathrm{SU}(2)$ "up to second order", which means
   $$g^*(t) g(t) = I + O(t^3) \; , \;\; \det(g(t) = 1 + O(t^3) \; .$$

(c) (*extra credit*) Show that $g(t) \in \mathrm{SU}(2)$ exactly. *Hint*: You can do it term by term calculating the Taylor series, but that proof is not interesting enough to be worth your time. If you know about differential equations, you can prove it by showing that $g(t)$ is the solution of the differential equation system

$$\frac{dg}{dt} = Lg \ , \quad g(0) = I \ .$$

(d) Assume the result of part (c). Show that the Lie algebra of $SU(2)$ consists of all matrices of the form (1).

(e) Show that if $L$ and $M$ are trace free, then the commutator, $[L, M] = LM - ML$ is trace free.

(f) Show that if $L$ and $M$ are skew hermitian, then the commutator is skew hermitian. Show that if $L \in \mathfrak{su}(2)$ and $M \in \mathfrak{su}(2)$, then $[L, M] \in \mathfrak{su}(2)$.

(g) The *Pauli spin matrices* are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Define corresponding elements of he Lie algebra $\mathfrak{su}(2)$

$$\begin{aligned} l_x &= \frac{1}{2i}\sigma_x \ , & l_y &= \frac{1}{2i}\sigma_y \ , & l_z &= \frac{1}{2i}\sigma_z \\ 2il_x &= \quad \sigma_x \ , & 2il_y &= \quad \sigma_y \ , & 2il_z &= \quad \sigma_z \ . \end{aligned}$$

Recall the order "x,y,z", which repeats to "y,z,x" and "z,x,y" (all part of "x,y,z,x,y,z"). Show that the Pauli matrices satisfy the "famous Pauli commutation relations"

$$\begin{aligned} [\sigma_x, \sigma_y] &= 2i\sigma_z \\ [\sigma_y, \sigma_z] &= 2i\sigma_x \\ [\sigma_z, \sigma_x] &= 2i\sigma_y \ . \end{aligned}$$

Conclude that the corresponding elements of $\mathfrak{su}(2)$ satisfy

$$\begin{aligned} [l_x, l_y] &= l_z \\ [l_y, l_z] &= l_x \\ [l_z, l_x] &= l_y \ . \end{aligned}$$

(h) An isomorphism between Lie algebras is a linear map $l \xrightarrow{A} Al = L$ that is one-to-one and onto, and preserves the commutator:

$$[Al, Am] = A[l, m] \ .$$

Show that the Lie algebra $\mathfrak{su}(2)$ is isomorphic to the Lie algebra of SO(3), which is $\mathfrak{so}(3)$.

In class we derived a set of generators of $\mathfrak{so}(3)$, which we called $L_x$, $L_y$, and $L_z$ and corresponded to infinitesimal rotation about the $x$, $y$, and $z$ axes respectively. A rotation by angle $t$ about the $x$ axis has the effect $x \to x$, and $(y, z) \to (\cos(t)y + \sin(t)z, -\sin(t)y + \cos(t)z)$. This action is represented in matrix form by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ \cos(t)y + \sin(t)z \\ -\sin(t)y + \cos(t)z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

This can be written

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to g_x(t) \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \quad g_x(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} .$$

The corresponding generator of the Lie algebra is

$$L_x = \frac{d}{dt} g_x(t) \Big|_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} .$$

In class, we calculated $L_y$ and $L_z$ and verified the commutation relations

$$\begin{aligned} [L_x, L_y] &= L_z \\ [L_y, L_z] &= L_x \\ [L_z, L_x] &= L_y \ . \end{aligned}$$

We also calculated the exponential maps (see next exercise) to be $e^{tL_x} = g_x(t)$, with formulas above, and similarly for $L_y$ and $L_z$.

(i) The *exponential map* defined in part (b) defines a homomorphism from the additive group $\mathbb{R}$ to $G$. The image is a one dimensional subgroup of $G$. Find the three one dimensional subgroups (formulas for the matrices as functions of $t$) for the two generators $l_y$, and $l_z$. Here is the answer for $l_x$. First you write the matrix Taylor series for $e^{tl_x}$. Then you put in the formula for $l_x$ and its powers. Then you collect the individual matrix elements (entries) as ordinary power series. Then you identify the functions these power series are Taylor series for – sines and cosines or exponentials.

$$
\begin{aligned}
g_x(t) &= I + tl_x + \frac{t^2}{2}l_x^2 + \frac{t^3}{6}l_x^3 + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} + \frac{t^3}{6}\begin{pmatrix} 0 & \frac{i}{8} \\ \frac{i}{8} & 0 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} 1 - \frac{t^2}{2\cdot 4} + \cdots & i(t - \frac{t^3}{6\cdot 8} + \cdots) \\ i(t - \frac{t^3}{6\cdot 8} + \cdots) & 1 - \frac{t^2}{2\cdot 4} + \cdots \end{pmatrix} \\
g_x(t) &= \begin{pmatrix} \cos(\frac{t}{2}) & -i\sin(\frac{t}{2}) \\ -i\sin(\frac{t}{2}) & \cos(\frac{t}{2}) \end{pmatrix} .
\end{aligned}
$$

The last step is guessing from the beginning of the Taylor series above. Therefore, the formula for $g_x(t)$ on the last line is only a conjecture that needs to be verified by a proof. The proof has four parts. First, you verify that $g_x(t)$ is the image of $\mathbb{R}^+$ (the additive group), which means $g_x(t_1 + t_2) = g_x(t_1)g_x(t_2)$ (why?). Second, you check that $g_x(t)$ is unitary. Third, you check that $\det(g_x(t) = 1$. Finally, you check that

$$
\frac{d}{dt}g_x(t)\Big|_{t=0} = l_x \ .
$$

(j) Show that the isomorphism of Lie algebras from part (h) does not "lift" to an isomorphism of the corresponding Lie groups. For this, take $l_z \in \mathfrak{su}(2)$ and the corresponding $L_z = Al_z \in \mathfrak{so}(3)$. Show that $e^{2\pi l_z} = -I$ in SU(2) but $e^{2\pi L_z} = I$ in SO(3). In other words, a full rotation in SO(3) corresponds to only a half rotation in SU(2).

3. This sequence of exercises continues Exercise 3 from Assignment 13. Some of the facts you need for these new exercises come from last week.

   (a) Show that if $a \neq 0 \bmod q$ there are numbers $\widehat{b}_j$ (which depend on $a$) so that

   $$
   \sum_{j=0}^{q-2} \widehat{b}_j \chi_j(n) = \begin{cases} 1 & \text{if } n = a \bmod q \\ 0 & \text{if } n \neq a \bmod q \ . \end{cases}
   $$

   Assume that $\chi_j(n) = 0$ if $n = 0 \bmod q$. Show that $\widehat{b}_0 \neq 0$. This is the coefficient of the trivial character.

   (b) Consider the function

   $$
   f(s) = \sum_{j=0}^{q-2} \widehat{b}_j \phi_j(s) \ .
   $$

   Show that $f(s) \to \infty$ as $s \downarrow 1$.

(c) Show that

$$f(s) = \sum_{p = a \bmod q} \frac{1}{p^s} + g(s) \ ,$$

where $g(s)$ is bounded as $s \downarrow 1$.

(d) Explain how all these exercises fit together to give a proof that there are infinitely many primes $p = a \bmod q$. This is a famous theorem of Dirichlet.