Honors Algebra II, Courant Institute, Spring 2021
http://www.math.nyu.edu/faculty/goodman/teaching/HonorsAlgebraII2021/HonorsAlgebraII.html
Check the class forum for corrections and hints

Assignment 5, due March 9 (before class starts).

Corrections

- March 5: Exercise 4(a) corrected to replace $f \notin R$ (which was silly) with $f \notin I$, and also for g.
- March 8: Exercise 4(a) corrected again to replace R by I again. Also, x is replaced by f in places there.

Instructions

- Do not hand in a rough draft. Copy or type answers neatly and clearly. Points may be deducted for writing that is sloppy, has excessive cross-outs, or is hard to read.
- State facts precisely in clear language or notation. Put assertions in logical order. State clearly what the hypotheses and conclusions. Put the steps of an argument in logical order, including definitions. Points may be deducted for an incorrectly stated argument even if you seen to understand it. Clear mathematical exposition is an important goal for the class.
- Learn the Greek letters used in math. Learn their mathematical names and write them clearly.

Assigned Exercises, to hand in

The first five are related to exercises people had trouble with in the quiz.

1. In the Gaussian integers $\mathbb{Z}[i]$ we can write $5 = (2+i)(2-i) = i(1-2i)(2-i)$. We also know the Gaussian integers are a unique factorization domain. The first factorization has two factors and the second has three. The first uses $2+i$ as one of the factors and the second one does not. Is this a contradiction?
2. Prove that $a = 2 + i$ is a prime element in the Gaussian integers.
3. You might conjecture that if R is a unique factorization domain and $I \subset R$ is a prime ideal, then R/I is a field. Check whether this is true for the example of $R = \mathbb{C}[x, y]$ (polynomial ring in two variables with complex coefficients) and $I_x = \{f \mid f(0, y) = 0 \text{ for all } y \in \mathbb{C}\}$.
4. An ideal $I \in R$ is *maximal* if it cannot be enlarged without becoming all of R. More precisely, if $I \subset J \subset R$ is a chain of ideals, then either $I = J$ or $J = R$.

(a) Show that a maximal ideal is prime. *Hint.* If I is not a prime ideal, then there is $fg \in I$ with $f \notin I$ and $g \notin I$. In that case $J = (f, I)$, which is the ideal generated by f and all of I, is an intermediate ideal. Part of this is giving a definition of (f, I) and showing it is an ideal.

(b) Show that I is a maximal ideal of an integral domain (no zero divisors) if and only if R/I is a field. *Hint.* If $f \in R$ has coset $(f) \in R/I$ that is not invertible, then (f, I) is an intermediate ideal, and conversely.

(c) Find a maximal ideal $J \subset \mathbb{C}[x, y]$ with $I_x \subset J$ (I_x is defined in Exercise 3.) and describe the field $\mathbb{C}[x, y]/J$.

5. Let $R = \mathbb{F}_p[x, y]$ be the polynomial ring in two variables with coefficients in the field $\mathbb{F}_p$. Define $I_x = \{f \mid f(0, y) = 0 \text{ for all } y \in \mathbb{F}_p\}$.

   (a) Show that I is an ideal in R.

   (b) Show that I is not a prime ideal.

6. Exercise 6.1 from Chapter 15. Before looking for an abstract proof, think about a polynomial $f \in \mathbb{R}[x]$ that has real or complex roots $z_1, \cdots, z_k$ with multiplicities $m_1, \cdots, m_k$. Write $f(x) = c \prod (x - z_j)^{m_j}$. What can you say about $f'(x)$ in terms of the data z_j and m_j?

7. Exercise 7.3 from Chapter 15. Before you start, imagine how long it would take to do this the brute force way by computing x^{13} for all $x \in \mathbb{F}_{13}^*$.

8. Exercise 7.6 from Chapter 15.

9. Exercise 7.7 from Chapter 15.

10. Exercise 7.13 from Chapter 15.