Assignment 8, due April 6 (before class starts).

Instructions

- Do not hand in a rough draft. Copy or type answers neatly and clearly. Points may be deducted for writing that is sloppy, has excessive cross-outs, or is hard to read.
- State facts precisely in clear language or notation. Put assertions in logical order. State clearly what the hypotheses and conclusions. Put the steps of an argument in logical order, including definitions. Points may be deducted for an incorrectly stated argument even if you seen to understand it. Clear mathematical exposition is an important goal for the class.
- Learn the Greek letters used in math. Learn their mathematical names and write them clearly.

Suggested exercise, not to hand in

1. Take $f \in \mathbb{Q}[t]$ to be $f(t) = (t^2+1)(t^2-2)$. Let $\mathbb{K}/\mathbb{Q}$ be a splitting field of f. For concreteness, you can take $\mathbb{K} \subset \mathbb{C}$, but this is not necessary. Carry out the construction described in Section 16.3 starting with the element $\beta_1 = i + \sqrt{2} \in \mathbb{K}$. What is the orbit of β_1 under the action of the permutation group on the roots of f? What is the polynomial $h(t) = \prod(t - \beta_j)$? If it is reducible, find the irreducible whose root is β_1.

Assigned Exercises, to hand in

1. Exercise 4.1(b) from Chapter 16. Extra credit: The following question is natural in view of this exercise. Suppose g and h are irreducible polynomials over $\mathbb{Q}$ and $f = gh$. Let $\mathbb{K}_f/\mathbb{Q}$ be the splitting field for f. Let $\mathbb{K}_g/\mathbb{Q}$ and $\mathbb{K}_h/\mathbb{Q}$ be splitting fields for g and h. Under what conditions (give conditions and/or counter-examples) is it true that the automorphism group of $\mathbb{K}_f/\mathbb{Q}$ the product of the groups for g and h?
2. Exercise 5.1 from Chapter 16. The field of rational functions (ratios of polynomials) is very important in the subject of *algebraic geometry* but not so much in this textbook. "determine explicitly" means describe what a function $u \in \mathbb{C}(t)$ looks like if it's in the fixed field.
3. Exercise 5.3 from Chapter 16 This fact seems almost trivial and the proof (once you see it) is short. I guess the author is interested in this fact because it's part of a hard and important theorem called the *Noether preparation theorem*.

4. Exercise 6.1 from Chapter 16
5. Exercise 6.3 from Chapter 16
6. The *orthogonal group* in n dimensions is the set of real $n \times n$ matrices Q with $Q^t Q = I$.
   (a) Show that the set of orthogonal matrices forms a group. This group is called $O(n)$.
   (b) Show that the following are equivalent
      i. $Q^t Q = I$
      ii. Q preserves the *inner product* of $\mathbb{R}^n$, which means that for $x_1, x_2 \in \mathbb{R}^n$ if $y_1 = Qx_1$ and $y_2 = Qx_2$, then $x_1^t x_2 = y_1^t y_2$.
      iii. Q preserves euclidean distance, which means that $\|y_1 - y_2\| = \|X_1 - X_2\|$. The euclidean length is $\|z\| = \sqrt{z^t z}$.
      iv. The columns of Q are orthogonal to each other, which means $q_j^t q_k = 1$ if $j \neq k$ (the term *orthogonal matrix* comes from this), and $\|q_j\| = 1$ for all j.

$$Q = \begin{pmatrix} | & | & \cdots & | \\ q_1 & q_2 & & q_n \\ | & | & \cdots & | \end{pmatrix} .$$

   (c) In three dimensions ($n = 3$), an orthogonal matrix is a *rotation* if $\det(Q) = 1$, and *orientation reversing* if $\det(Q) = -1$. Show that any orthogonal matrix is rotation or orientation reversing.
   (d) Consider the complex eigenvalue problem for Q, which is $Qz = \rho z$ with $\rho \in \mathbb{C}$ and $u, v \in \mathbb{C}^3$. Show that the eigenvalues of an orthogonal rotation in three dimensions satisfy one of the following
      i. $\rho_1 = \rho_2 = \rho_3 = 1$, so $Q = I$
      ii. $\rho_1 = \rho_2 = -1$, and $\rho_3 = 1$.
      iii. $\rho_2 = \overline{\rho}_1$ and $\rho_3 = 1$.
   (e) In case (iii), show that $\rho_1 = e^{i\theta}$ for some real angle θ, and Q corresponds to a rotation by angle θ about the real eigenvector corresponding to eigenvalue $\rho_3 = 1$. Show that case (ii) corresponds to $\theta = \pm\pi$. An *axis* is a line through the origin. The axis that contains all eigenvectors of Q with eigenvalue $\rho_3 = 1$ is the *axis of rotation* of Q. Each rotation except $Q = I$ has a unique axis of rotation.
   (f) Let $H \subset O(3)$ be the subgroup that fixes the "$z-$axis" (which is the set of points of the form $(0, 0, a)$). Describe matrices $Q \in H$ in terms of a 2×2 rotation matrix by angle θ.

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

(g) Show that H is not a normal subgroup of $O(3)$ (which is how this exercise fits into the material for the week).

(h) The set of all axes in three dimension is the *real projective space*, RP^2. Show that there is a one to one correspondence between left cosets in $H\backslash O(3)$ and elements of RP^2. Whenever $H \subset G$ is a subgroup, a left coset is an equivalence class of the form hg, where $g \sim hg$ for $h \in H$ and $g \in G$. *Left cosets* correspond to the equivalence relation $g \sim gh$. These equivalence relations are different if H is not normal. If H is normal, left cosets are right cosets, which is why we write G/H for the quotient group even if it is defined using left cosets. The fact that the coset space $RP^2 = H\backslash O(3)$ is not a group is associated to the fact that $H \subset O(3)$ is not a normal subgroup.

(i) (not to hand in, because it's geometry, not algebra). The real projective space RP^2 is a two dimensional surface something like the unit sphere in $\mathbb{R}^3$, which is called the "2−sphere", S^2. One difference is that it is impossible to *embed* RP^2 in $\mathbb{R}^3$ without self intersections. This will be clear if you do a web search on "real projective space model".