Practice for the Quiz, February 25.

Quiz Instructions and information

- You must do the quiz while on zoom with your camera on at all times. Upload your answers at the end of the quiz.
- The quiz will be closed book, closed notes, etc. You may not use any resources during the quiz, except ...
- You may prepare and use a "cheat sheet", which is one US standard size ($8\frac{1}{2}'' \times 11''$) piece of paper, front and back. Please upload the cheat sheet with your quiz answers, if you use one.
- Please write as clearly and neatly as possible in a quiz situation. If you scan or photograph a handwritten paper (the most common mode), please do that as well as possible in the quiz setting.
- You will be graded on clarity as well as mathematical correctness. You don't have to use full sentences in each case, but what you write should be grammatical and use mathematical terms and notation correctly. You may use scratch paper that you don't hand in to organize your thoughts. Reasoning is as important as the answer in a theory class like abstract algebra.
- You will get 25% credit for any question or question part that you leave blank. You may lose points for a wrong answer, even if you also give a correct answer. Cross out anything you think is wrong.
- If you have a question during the quiz, please communicate with me on chat in the zoom session. Make sure the chat goes only to me and not the everyone in the session.

Questions

The actual quiz will be much shorter than this.

1. In each case, give an example or prove that there are no such examples.
   (a) A ring R with $0 \neq 1$ (additive identity $\neq$ multiplicative identity) that has no ideals except $I = \{0\}$ and $I = R$.
   (b) A ring R where every ideal is a principal ideal.
   (c) A ring and an ideal that is not a principal ideal.
   (d) A ring where the only principal ideals are $I = \{0\}$ and $T = R$.
   (e) A ring with an element $x \neq 0$ and $x^2 = 0$.

(f) A field with 25 elements so that $5x = 0$ for all x. (5 is not an element of the field, the definition is $5x = x + x + x + x + x$.

(g) An infinite field with $5x = 0$ for all x.

(h) A field with 50 elements and $5x = 0$ for all x.

(i) A ring R that contains $\mathbb{Z}$ as a subring so that p is irreducible in $\mathbb{Z}$ but not in R.

2. Find the prime factorization of $7i$ in the Gaussian integers $\mathbb{Z}[i]$

3. Show that there is an $x \in \mathbb{Z}$ with $x^2 \equiv -1 \bmod p$ depending on whether $p - 1$ is a multiple of 4 (i.e. $p \equiv 1 \bmod 4$) or not ($p \equiv 3 \bmod 4$). Hint: $x = g^k$ for some k where g is a generator of $\mathbb{F}_p^*$.

4. Show that $f(x) = 1 + x - 3x^3$ has no rational roots. *Hint.* Integer roots?

5. In each case either describe a field injection $\mathbb{F} \mapsto \mathbb{K}$ or prove that there no such injecction

   (a) $\mathbb{Q}[i] \subset \mathbb{C} \mapsto \mathbb{R}$

   (b) $\mathbb{Q}(x) \mapsto \mathbb{C}$

   (c) $\mathbb{R}(x) \mapsto \mathbb{C}$

   (d) $\mathbb{Q}[\alpha] \subset \mathbb{C} \mapsto \mathbb{R}$, where $\alpha = 2^{\frac{1}{3}} e^{2\pi i/3}$ and $2^{\frac{1}{3}}$ is the positive real cube root.

   (e) $\mathbb{F}_{29} \mapsto \mathbb{F}_{57}$

6. Find an integer n with $n \equiv 2 \bmod 5$, $n \equiv 3 \bmod 13$, and $n \equiv 4 \bmod 23$. Is n unique? What is the smallest positive such n? *Hint.* If you want to do this in a reasonable amount of time, you should not use trial and error.

7. 521 and 523 are prime integers. Which of them factors in $\mathbb{Z}[i]$?

8. A field $\mathbb{F}$ has *characteristic* p, where p is an integer, if $p \cdot x = x + x + \cdots + x = 0$ (adding p terms of x) for all $x \in \mathbb{F}$. [A field that does not have characteristic p for any p is said to have *characteristic zero*.]

   (a) Show that p must be prime.

   (b) Show that the finite field $\mathbb{F}_p$ has characteristic p.

   (c) Show that if $p \cdot x = 0$ for any $x \neq 0 \in \mathbb{F}$ then $\mathbb{F}$ has characteristic p.

   (d) Show that if $\mathbb{F} \subset \mathbb{K}$ is a field extension, and $\mathbb{F}$ has characteristic p, then $\mathbb{K}$ also has characteristic p.

   (e) Give an example of an infinite field of characteristic p.

9. Explain the proof that $\mathbb{Q}[x, y]$ is a principal ideal domain.