Practice and information for the Final exam

- The exam will be Tuesday, May 17, 11 am – 12:50 pm in person in room 512.
- The exam will cover all the material from the semester, including material covered on the quiz and midterm and material in the assignments not from the book.
- The final exam will be closed book, closed notes, etc. You may not use any resources during the exam, except ...
- You may bring and use a "cheat sheet", which is one US standard size ($8\frac{1}{2}'' \times 11''$) piece of paper, front and back. Please upload the cheat sheet with your quiz answers, if you use one.
- Please write as clearly and neatly as possible in a quiz situation. If you scan or photograph a handwritten paper (the most common mode), please do that as well as possible in the quiz setting.
- You will be graded on clarity as well as mathematical correctness. You don't have to use full sentences in each case, but what you write should be grammatical and use mathematical terms and notation correctly. You may use scratch paper that you don't hand in to organize your thoughts. Reasoning is as important as the answer in a theory class like abstract algebra.
- You will get 25% credit for any question or question part that you leave blank. You may lose points for a wrong answer, even if you also give a correct answer. Cross out anything you think is wrong.
- These questions are intended as review aids as well as practice. Some of them are too long for the actual exam but suggest material you should try to be familiar with.

Questions

1. Let A be the $n \times n$ "real matrix" (matrix with real entries) that has all

entries $a_{jk} = 0$ except $a_{k,k-1} = 1$ and $a_{k,k+1} = 1$. That is

$$A = \begin{pmatrix} 0 & 1 & 0 & & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & \vdots \\ 0 & 1 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & & \ddots & 1 \\ 0 & \cdots & & 0 & 1 & 0 \end{pmatrix} .$$

Let $p_n(x) = \det(xI - A)$ be the characteristic polynomial of A. Use the cofactor expansion along the first row or column or last row or column to show that

$$p_{n+1}(x) = xp_n(x) - p_{n-1}(x) .$$

[If you replace A with $\frac{1}{2}A$, the characteristic polynomials are called *Chebychev* polynomials. They are important in numerical analysis.]

2. Consider the algebraic number ring $\mathbb{Z}[i]$ as a lattice in $\mathbb{C}$. Give examples L of the following proper sub-lattices ("proper" means $L \neq \mathbb{Z}[i]$ and not $L \neq \{0\}$).

    (a) L is a prime ideal of the form $L = a\mathbb{Z}[i]$, with $a \in \mathbb{Z}$.

    (b) L is a prime ideal of the form $L = \alpha\mathbb{Z}[i]$, with $\alpha \notin \mathbb{Z}$.

    (c) L is an ideal but not a prime ideal of the form $L = n\mathbb{Z}[i]$, with $n \in \mathbb{Z}$.

    (d) L is an ideal but not a prime ideal of the form $L = \alpha\mathbb{Z}[i]$, with $\alpha \notin \mathbb{Z}$.

    (e) L is a full and not an ideal.

    (f) L is not full.

3. A *euclidean domain*, R, is defined by a "size function" $s(x)$ so that (among other properties) for any $x \neq 0$ and any y there are q and r so that $y = qx + r$ with $s(r) < s(x)$.

    (a) Use the fact that $(1 + i\sqrt{5})(1 - i\sqrt{5}) = 2 \cdot 3$ to show that $\mathbb{Z}[i\sqrt{5}]$ is not a euclidean domain.

    (b) The ring $\mathbb{R}[X]$ has the identity $(X^2+3X+2)(X+3) = (X+1)(X^2+5X+6)$. Does this imply that $\mathbb{R}[X]$ is not a euclidean domain?

4. Let $\xi \in \mathbb{R}$ be defined by $\xi = \left(1 + 2^{1/3}\right)^{1/5}$. Define the field $\mathbb{F} = \mathbb{Q}[\xi]$. Show that $\sqrt{n} \notin \mathbb{F}$ if $n \in \mathbb{Z}$ is square free.

5. Let $\mathbb{F} \subset \mathbb{C}$ be the splitting field over $\mathbb{Q}$ of the polynomial $f(X) = X^3 - 2$. Is there an $\alpha \in \mathbb{F}$ and a square free $n \in \mathbb{Z}$ with $\alpha^2 = n$? [If $n > 0$, you would just say $\alpha = \sqrt{n}$. I think $n = 0$ is should not be called square free because $n = n^2$.]

6. Let $\mathbb{F}/\mathbb{Q}$ be an *algebraic number field*, which just means that $\mathbb{F}$ is a finite degree extension of $\mathbb{Q}$. Let $R \subset \mathbb{F}$ be the set of algebraic integers (elements that satisfy monic integer polynomial equations over $\mathbb{Z}$). Let A and B be ideal in R. We say (it's in the book) that $A|B$ if there is an ideal C so that $AC = B$ (ideal multiplication). Let $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{Q}$ be an element of the Galois group. Let $A^\sigma = \{\sigma(\alpha) \mid \alpha \in A\}$. Show that $A^\sigma|(n)$ if $A|(n)$, where $(n) \subset R$ is the principal ideal generated by $n \in \mathbb{Z}$.

7. Let $\mathbb{K}/\mathbb{F}$ be a finite degree field extension. The degree of $\alpha \in \mathbb{K}$ is

$$\deg_{\mathbb{F}}(\alpha) = \min\{\, \deg(f) \mid f(\alpha) = 0\ ,\quad f \in \mathbb{F}[X]\,\}\,.$$

   (a) Show that $\deg_{\mathbb{F}}(\alpha) = [\mathbb{K} : \mathbb{F}]$ if and only if $\mathbb{K} = \mathbb{F}[\alpha]$.

   (b) Show that if $\alpha = \sigma(\alpha)$ for some $\sigma \in G = \mathrm{Gal}(\mathbb{K}/\mathbb{F})$, then $\deg_{\mathbb{F}}(\alpha) \mid [\mathbb{K} : \mathbb{F}]/\mathrm{ord}(\sigma)$. The *order* of σ is $\mathrm{ord}(\sigma)$, which is the number of distinct elements σ^k.

8. A primitive 8^{th} root of unity in $\mathbb{C}$ is

$$\zeta_8 = e^{2\pi i/8} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1+i}{\sqrt{2}}\ .$$

   (a) Show that the elements $\left\{1, i, \sqrt{2}, i\sqrt{2}\right\}$ form a basis for $\mathbb{Q}[\zeta_8]$ over $\mathbb{Q}$.

   (b) Find the 4×4 rational matrix, A, that expresses the action of $v \to (1 + i\sqrt{2})v$ (for all $v \in \mathbb{Q}[\zeta_8]$) in this basis.

   (c) Use the determinant cofactor expansion to find the characteristic polynomial $f(x) = \det(xI - A)$.

   (d) Verify that $f(\alpha) = 0$ where $\alpha = 1 + i\sqrt{2}$.

9. Describe the following quotient rings. How many elements does it have? If it is a field, describe the field. If not, give a zero-divisor.

   (a) $\mathbb{Z}[i]/(3)$

   (b) $\mathbb{Z}[i]/(5)$

   (c) $\mathbb{Z}[i\sqrt{5}]/(2)$