Honors Algebra II, Courant Institute, Spring 2022
http://www.math.nyu.edu/faculty/goodman/teaching/HonorsAlgebraII2022/HonorsAlgebraII.html
Check the Brightspace discussion page for corrections and hints

Practice for the Quiz, February 18.

Quiz Instructions and information

- The quiz will take 40 minutes.
- The quiz will be given in recitation on Friday. It will not be given online.
- The quiz will be closed book, closed notes, etc. You may not use any resources during the quiz, except ...
- You may bring and use a "cheat sheet", which is one US standard size ($8\frac{1}{2}'' \times 11''$) piece of paper, front and back. Please upload the cheat sheet with your quiz answers, if you use one.
- Please write as clearly and neatly as possible in a quiz situation. If you scan or photograph a handwritten paper (the most common mode), please do that as well as possible in the quiz setting.
- You will be graded on clarity as well as mathematical correctness. You don't have to use full sentences in each case, but what you write should be grammatical and use mathematical terms and notation correctly. You may use scratch paper that you don't hand in to organize your thoughts. Reasoning is as important as the answer in a theory class like abstract algebra.
- You will get 25% credit for any question or question part that you leave blank. You may lose points for a wrong answer, even if you also give a correct answer. Cross out anything you think is wrong.

Questions

This variety of questions should help you study. The actual quiz will have fewer problems.

1. In each case, give an example or prove that there are no such examples.
   (a) A ring R with $0 \neq 1$ (additive identity $\neq$ multiplicative identity) that has no ideals except $I = \{0\}$ and $I = R$.
   (b) A ring R where every ideal is a principal ideal.
   (c) A ring and an ideal that is not a principal ideal.
   (d) A ring where the only principal ideals are $I = \{0\}$ and $T = R$.
   (e) A ring with an element $x \neq 0$ and $x^2 = 0$.

(f) A finite ring that is not a field.

(g) An infinite ring that is not a field with infinitely many units.

2. Show that $\zeta_3 - 1$ and $\zeta_3^2 - 1$ are associates in the ring $\mathbb{Z}[\zeta_3]$, where $\zeta_3 = e^{2\pi i/3} \in \mathbb{C}$. Find a factorization of 3 into a product of irreducible elements in $\mathbb{Z}[\zeta_3]$.

3. Use the little Fermat theorem to show that if x, y, and z are integers with $x^4 + y^4 = z^4$, then x or y (or both) are multiples of 5. (Exercise 1 from the Motivation notes. In general, understand the solutions to the homework exercises.)

4. Find an factorization into irreducible polynomials of $f(X) = X^5 - 1$ in $\mathbb{Z}[X]$ and $\mathbb{F}_5[X]$.

5. Let R be a Noetherian ring. Show that if $a \in R$ is not a unit then there is a field $\mathbb{F}$ and a homomorphism $\phi : R \to \mathbb{F}$ with $a \in \mathrm{ket}(\phi)$. Is this possible if $a \in R$ is a unit? *Note*, it is not necessary that R be Noetherian, but the proof otherwise involves the axiom of choice.

6. What is $\phi(128)$? Here, ϕ is the Euler "totient" function.

7. Is there an integer x with $x^2 \equiv 180 \bmod 181$? *Note*, 181 is prime.

8. Show that $\mathbb{C}[X, Y]$ is not a principal ideal domain.

9. Show that the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\zeta_3]$ are not isomorphic.