Calculus I, Fall 2004, sections 4, 5, and 6. Courant Institute of Mathematical Sciences, New York University.

Practice for the Final

Most of the questions on the final will be closely related to questions here. The final exam itself will have 8 questions with at most 4 parts per question. You may bring a "cheat sheet", one piece of paper with anything you like written on it. There will be no calculators on the final. Please use a mechanical pencil or pen. You will receive one or two points for no answer on any question and we will take off for wrong answers.

1. Find the following limits.

   (a) $\lim_{x\to 2} \dfrac{x^2-4x+3}{x^2-1}$

   (b) $\lim_{x\to 1} \dfrac{x^2+2x-3}{x^3-3x+2}$

   (c) $\lim_{x\to\infty} x\tan\left(\dfrac{4}{x}\right)$

   (d) $\lim_{x\to\infty} \dfrac{1+3x-\cos(2x)}{x}$

   (e) $\lim_{x\to 0} \dfrac{e^{-x}-1}{x}$

   (f) $\lim_{x\to\infty} x\ln\left(1+\dfrac{1}{x}\right)$

2. Find the following derivatives.

   (a) $\dfrac{d}{dx}\left[(x+1)^2\right]$. Use the definition of the derivative.

   (b) $\dfrac{d}{dx}\sqrt{2x-3}$. Use the definition of the derivative.

   (c) $\dfrac{d}{dt}\left(\dfrac{t^2+4t+3}{t+1}\right)$. Simplify your answer.

   (d) $\dfrac{du}{dv}$ from the equation $vu+v^3u^2=\sin v$

   (e) $\dfrac{d}{dt}t^2\ln(1+t^3)$.

   (f) $\dfrac{d^2}{dx^2}\left(\sec x-\cos x\tan^2 x\right)$. Simplify your answer.

   (g) $\dfrac{d^2}{dt^2}\sin(u(t))$ when $t=2$, given that $u(2)=\pi/6$, $u'(2)=3$, and $u''(2)=4$.

   (h) $\dfrac{d}{dx}xe^{-2x}$.

(i) $\dfrac{d}{dx}\cos(x)^x$.

(j) $\dfrac{d}{dy}\log_y(10)$. Hint: Express $\ln(10)$ in terms of $\log_y(10)$, then differentiate that formula.

(k) $\dfrac{d}{dt}e^{\sqrt{t}}$.

(l) $\dfrac{d}{dt}\displaystyle\int_0^{t^2}\sin\sqrt{x}dx$.

3. A subway car is moving on a straight section of track with the distance formula $at^2 + bt + c$, where t is measured in seconds from the time you start watching. After the first ten seconds, the car has moved 12 feet and is going 13 feet per second. How far has the car gone after another ten seconds? What is your acceleration at that time?

4. A spherical balloon is pumped full of air at 250 cc/s. How fast is its surface area changing when it has been filled with 5,000 cc of air?

5. If f is the focal length of a convex lens and an object is placed at a distance p from the lens, then its image will be at a distance q from the lens, where f, p, and q are related by the *lens equation*

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$

Find the rate of change of p with respect to q.

6. The circumference of a sphere is measured to be 84 cm with a possible error of .05 cm. What is the maximum error in the calculated surface area? (The surface area of a sphere of radius r is $4\pi r^2$.)

7. Use the first derivative approximation to estimate:

(a) $\sqrt{.9}$.

(b) $\cos(\dfrac{\pi}{4} + \dfrac{\pi}{20})$.

(c) the value of x so that $xe^x = .2$. Hint: let $x = u(y)$ be the inverse function of $y = xe^x$. We are looking for $u(.2)$. A simpler way to think of it (thought the calculations are the same) is to make a linear approximation to $f(x) = xe^x$ that is valid when x is small, then find the x value so that the approximation is .2.

(d) $\displaystyle\int_0^1 \ln(1 + .3\sin(x))dx$. Hint: make an approximation to $\ln(1 + y)$ when y is close to zero using the value of $\ln(1)$ and the derivative at $y = 1$.

8. Draw careful graphs of the following functions. Include all important features of the graph which may include: intercepts, roots, asymptotes, critical values, and inflection points.

   (a) $f(x) = \dfrac{\sin x}{1 - \sin x}$ for $x \in (-\pi, \pi)$

   (b) $g(x) = x^5 - 3x^2$

   (c) $h(x) = x\sqrt{x-1}$

   (d) $k(x) = \dfrac{x^2}{x^2 - 4}$

9. Find the equation of the line that is normal to the curve $y = x^2$ and goes through the point $(2, \frac{1}{2})$. (Normal to the curve means normal to the tangent line of the curve).

10. A power line is needed to connect a power station on the shore of a river to an island which is 3 miles downstream and 2 miles offshore. Find the minimum cost for such a line given that it costs \$60,000 per mile to lay wire underwater and \$40,000 per mile to lay wire under ground.

11. Calculate

    (a) $\displaystyle\int_{-1}^{1} x^2 dx.$

    (b) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{1+3t}} dt.$

    (c) $\displaystyle\int_{\pi/2}^{\pi} \cos\left(\frac{t}{2}\right) dt.$

    (d) $\displaystyle\int_{1}^{2} x \ln\left(1 + x^2\right) dx.$ Hint: $\int \ln(u)du = u\ln(u) - u.$

    (e) $\displaystyle\int_{-1}^{1} \sin\left(t^3\right) dt.$ Hint: You cannot find the indefinite integral, but the answer is obvious if you draw a graph.

    (f) $\displaystyle\int_{-\pi}^{\pi} x \sin(x) dx.$

    (g) Find $F(x)$ with $F'(x) = \dfrac{e^{x/2}}{1 + e^{x/2}}$.

    (h) Find $F(x)$ with $F'(x) = \dfrac{e^{x/2}}{1 + e^{x}}$. Hint: Note that $e^x = \left(e^{x/2}\right)^2$.

    (i) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{1-x}{\sqrt{1-x^2}} dx.$ Hint: Break the integral into two integrals and work them in different ways.

    (j) $\displaystyle\lim_{R\to\infty} \int_{0}^{R} e^{-at} dt.$

12. The number of bacteria in a dish is growing exponentially. We cannot measure the number of bacteria directly, but we can determine how fast the number is changing because cell division releases a chemical we can detect. The rate of growth is 3×10^6 bacteria per hour initially and 4.5×10^6 bacteria per hour one hour later.

    (a) What was the initial population?

    (b) How many new bacteria were "born" during the first hour? Hint: more than 3×10^6 because the rate of growth is increasing. 3×10^6 is the number that would be born if the rate of growth would not change during the hour.

13. The number of phone calls per minute in response to an annoying incident on TV had the form $f(t) = Ate^{-rt}$.

    (a) Make a rough sketch of this function for $t > 0$ assuming that A and r are positive. The sketch should show where $f(t)$ is zero and how f increases and decreases.

    (b) Minute 60 had 100 calls, which was more calls than any other minute. How many calls were received in total? Interpret this as saying that $f(60 \text{ minutes}) = 100$ calls/minute.

14. The part of the graph of $y = 1 - x^2$ above the x axis is rotated about the x axis to form a football-shaped three-dimensional object. Find the volume of this object.