Assignment 6. Variance reduction methods

Given December 10, due at the end of finals week.

Objective: To explore various variance reduction techniques in a model problem.

The diffusion equation $\partial_t u = \frac{1}{2}\partial_x^2 u$ has some polynomial explicit solutions that can be used to test variance reduction methods. For example, if $u(x,0) = x^4$, then $u(x,t) = x^4 + a(t)x^2 + b(t)$, where the coefficients, a and b can be found by substituting into the diffusion equation and solving the resulting ordinary differential equations. From general theory, we know that

$$u(x, T-t) = \mathbf{E}_{x,t}\left[X(T)^4\right] \quad , \tag{1}$$

where $\mathbf{E}_{x,t}$ means "expected value starting from x at time t, and $dX = dZ$.

1. Take $T = 3$, $x = 1$, and various values of t between zero and three. Estimate u from (1) using Monte Carlo and the method of antithetic variates. Note how the variance reduction properties of antithetic variates depend on the time to expiration. Do you have any idea why this might be true?

2. Now introduce "stochastic volitility" by replacing $dX = dZ$ by

$$dX = \sigma(t)dZ_1 \quad , \tag{2}$$

$$d\sigma = \sigma_0 dZ_2 \quad , \tag{3}$$

where Z_1 and Z_2 are independent Brownian motions. Take $x = 1$, $t = 0$, and $T = 1$, $\sigma(0) = 1$, and various values of the constant σ_0. Estimate (1) using Monte Carlo and the technique of control variates, where the control variate is the process with the $\sigma_0 = 0$, which is known analytically (see above). Discuss how the effectiveness of this depends on σ_0