Assignment 2.

Given September 17, due October 1.

Objective: Basic familiarity with finite differences.

A certain illegal organization establishes an illegal interest rate, $R(t)$. This rate satisfies the stochastic differential equation

$$dR = \alpha\sqrt{R}dt + \sigma\sqrt{R}dZ \quad .$$

Here, t and is measured in years and α and σ are measured accordingly. Presently, $R = 10$. The organization also indulges in gambling. It will sell a wager that pays one dollar after a year if $|R(1) - 10| \leq A$. It will not pay if $|R(t) - 10| \geq 5$ at any time $0 \leq t \leq 1$ (a "knockout" feature in its language and on Wall street).

(1) Write the backward equation to compute the present expected value of the wager. Neglect discounting factors. Take $A = 2$, $\alpha = 1$, and $\sigma = 2$. Solve the backward equation using the forward Euler finite difference method with various values of Δr and Δt but with the ratio $\Delta t/\Delta r^2$ fixed.

- **a.** Do a numerical convergence study using your code to determine the order of accuracy of your method. Does it agree with the theoretical order of accuracy?
- **b.** What is a reasonable Δr if you want .1% accuracy in the answer.

(2) Write the forward equation for the probability density of $R(t)$. Solve it by forwards Euler. Use the methodology of part 1 to choose a reasonable step size. Compute the expected value of the wager by integrating the probability density at time $t = 1$ against the payout.

(3) We want to evaluate the sensitivity $\Lambda = \partial_\sigma f$.

- **a.** Use the code for part 1a and finite differences to evaluate Λ for $\sigma = 2$.
- **b.** Write the backward equation satisfied by $\partial_\sigma f$ and solve it directly. Does the result agree with part a? What is its order of accuracy?

(4) For the same values of σ and α, compute and plot the expected value as a function of A in the range $1 \leq A \leq 3$. Try to adapt the two methods in part 3 to compute Ω (the last possible Greek), the derivative with respect to A.

Note on programming. This is partly an exercise in software engineering. If you code from scratch for each part, it will be tedious. Plan the work so that routines (such as that for convergence analysis) can be used over.