Complex Variables II
Assignment 1

"The book" is *Basic Complex Analysis* by Marsden and Hoffman. References to page or propositions etc., are references to thing in "the book". "Alfors" refers to *Complex Analysis* by Lars Alfors.

This assignment has multiple purposes, including the obvious ones of helping you think through the material and giving me a basis for a grade. It's also a placement test that I can use to see were you are in complex analysis. This is to help me pitch future assignments. Hopefully they won't be too easy or too hard or too long or too short.

Please write or print your answers on paper and bring it to class on January 31.

1. (*This is an exercise in topological reasoning of the kind often used in Complex Analysis. There is reasoning like this in the book, but it's a bit muddled.*)

   Definition. Let a and b be in the open set A, then $a \sim b$ if there is a piecewise differentiable path $\gamma(t)$ defined for $0 \leq t \leq T$ with $\gamma(0) = a$ and $\gamma(T) = b$ and $\gamma(t) \in A$ for $0 \leq t \leq T$. Note that Definition 1.4.11 (in the book) asks for γ to be differentiable, not piecewise differentiable, which is unnecessary in applications and makes proofs more complicated. For example, the book's "proof" of Proposition 1.4.15, which is essentially part (a) below, is incomplete on this point. It is easy to extend a path from z_0 to z using piecewise differentiable paths. But it takes more work to make the path differentiable at z_0, a point the book ignores.

   Definition. The path connected component of $a \in A$ is the set of points $b \in A$ with $b \sim a$. An open set A is path connected if it has a unique path connected component.

   Definition. The open set A is disconnected if there are disjoint open sets $B \subset \mathbb{C}$ and $C \subset \mathbb{C}$ so that $A = B \bigcup C$. if $a \sim b$ for any $a \in A$.

   (a) Show that a path component of an open set in $\mathbb{C}$ is an open set in $\mathbb{C}$.

   (b) Show that an open set in $\mathbb{C}$ is path connected if and only if it is connected.

   (c) Give an example of a bounded open set in $\mathbb{C}$ that has infinitely many path connected components.

   (d) *This is footnote 6 on page 48,* Consider the set $K \subset \mathbb{C}$ consisting of the points of the form $x + i\sin(\frac{1}{x})$ for $x > 0$ together with the interval of points it with $|t| \leq 1$. Show that K is closed and connected but not path connected. *Connected* means that if $K \subset A \bigcup B$ with A and B open sets, then either $K \subset A$ or $K \subset B$ or $A \bigcap B$ is not empty. To show K is not path connected, try to show there is no continuous function (let alone piecewise differentiable) γ with $\gamma(0) = 0$ and $\gamma(1) = 1 + i\sin(1)$. To show K is connected, suppose K is contained in a union of two disjoint open sets. One of those sets must contain the interval $\{it\}$ because the interval is connected. One of the sets must contain the graph $\left\{x + i\sin(\frac{1}{x})\right\}$ for the same reason.

2. The *inverse function theorem* for a real valued function of a real variable $u(x)$ says that if u differentiable enough and $u'(0 \neq 0$, then u is invertible near 0. "Invertible near zero" means that there are numbers $r > 0$ and $s > 0$ so that if $|a| < r$ then there is a unique x with $|x| < s$ so that $u(x) = u(0) + a$. The inverse function theorem for complex analytic functions is similar. If f is analytic in a neighborhood of (disk containing) z_0 and $f'(z_0) \neq 0$, then there are r and s so that there is a unique z with $|z - z_0| < s$ so that $f(z) = w$, if $|w - f(z_0)| < r$. Denote this local inverse function by $g(w) = z$, with $f(g(w)) = w$. The analytic inverse function theorem for analytic functions says says that g is analytic in a neighborhood of $w_0 = f(z_0)$. This is Theorem 5.1.2 in the book The proof of part (i) in the book relies the real inverse function theorem from real analysis for multi-variate functions.

   Here is a proof of the inverse function theorem for analytic functions that uses contour integration instead. Suppose $f(z)$ is analytic for z in some open set Ω and that $f'(z_0) \neq 0$ for some $z_0 \in \Omega$. We write $D(a, \rho)$ for the open disk in the complex plane of radius ρ centered at a.

   (a) (*uniqueness*) Show that there is a $\rho > 0$ so that if $z_1 \in D(z_0, \rho)$ and $z_2 \in D(z_0, \rho)$ with $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$.

   (b) (*contour*) Define the contour integral

   $$I(w) = \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta) - w} d\zeta \ .$$

   Show that if $f'(z_0) \neq 0$, then there is a contour $\gamma(t) = z_0 + Re^{it}$, $0 \leq t \leq 2\pi$, so that the integrand is analytic on γ, $I(w_0) = 1$, and $I(w)$ is an analytic function of w. (This shows $I(w) \neq 0$ for w close enough to w_0. A more careful analysis shows $I(w) = 1$ for w close to w_0, but that is not needed for this proof.)

   (c) (*existence*) Conclude from part (b) that if w_1 is close enough to w_0, there is at least one z_1 close to z_0 with $f(z_1) = w_1$. (Please carry this out explicitly even though it is a corollary of Rouché's theorem.)

   (d) (*necessity*) What conclusions of parts (a) and (b) are false for $f(z) = z^2$, $z_0 = 0$?

(e) (*analyticity*) Consider the modified integral

$$J(w) = \frac{1}{2\pi i} \int_\gamma \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta \ .$$

Show that if R is small enough and w is close enough to w_0, then $J(w) = z$ has $f(z) = w$.

3. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ipx}}{1 + x^4} \, dx \ .$$

Here p is a real *wave number*. Justify the contour manipulations in detail, this is the point of this exercise. Note that the "completion" of the real part of the contour to the upper or lower half of the complex plane depends on the sign of p.

4. (*Practice using the reasoning behind the Schwarz lemma*) Suppose $f(z)$ is analytic for $|z| < 1$ and continuous on $|z| \leq 1$ (the usual hypotheses), and that $f(0) = 0$ and $f'(0) = 0$. Define

$$M = \max_{|z|=1} |f(z)| \ .$$

Show that $|f(z)| \leq M |z|^2$. Show that if equality holds for any $|z| < 1$ then $f(z) = cz^2$ for some c with $|c| = M$.

5. (*This is a technical lemma for Exercise 6*) Apply Exercise 4 to show there is a C with

$$|e^z - (1 + z)| \leq C |z|^2 \ , \quad \text{if } |z| \leq 1 \ .$$

6. Define

$$f(z) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t-z)^2} \, dt \ .$$

Contour integration shows that $f(z)$ is independent of z (See Figure 4.3.10 in the book.), but this exercise asks you to verify it directly. Show from first principles and direct estimates (inequalities) that $f(z)$ is defined for each $z \in \mathbb{C}$ and that the following limit exists

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \ .$$

You may assume the convergence of real integrals of the form

$$\int_{-\infty}^{\infty} e^{-at^2 + bt} \, dt \ , \quad \text{with } a > 0 \ , \quad \text{and } b \in \mathbb{C} \ .$$

Show that $f(x)$ is independent of x for $x \in \mathbb{R}$ and conclude that $f(z)$ is independent of z for $z \in \mathbb{C}$.