Complex Variables II
Assignment 11

1. The two dimensional *complex projective space*, as a set, is defined to be the set of equivalence classes of non-zero points $Z \in \mathbb{C}^3$. We will call it $\mathbf{CP}^2$. We use capital letters such as $Z = (Z_0 : Z_1 : Z_2)$ for elements of $\mathbb{C}^3$, "point" letters like p and q for elements of $\mathbf{CP}^2$, and lower case letters such as $z = (z_1, z_2)$ for local coordinates on $\mathbf{CP}^2$. The equivalence relation is $Z \sim W$ if there is an $a \in \mathbb{C}$ so that $Z = aW$ in $\mathbb{C}^3$. We require $Z \neq 0$ and $W \neq 0$, which implies $a \neq 0$. We write $[Z]$ for the equivalence class of Z:
$$[Z] = \{\, aZ \text{ for all } a \in \mathbb{C}\} \ .$$
A point $p \in \mathbf{CP}^2$ is an equivalence class. There are three basic coordinate patches $U_k \subset \mathbf{CP}^2$, with U_k being the family of equivalence classes $[Z]$ with $Z_k \neq 0$. For example, each $p \in U_0$ is represented by an equivalence class $[Z]$ with $Z = (1 : z_1 : z_2)$. Similarly, a point $q \in U_1$ is an equivalence class of the form $[(w_1 : 1 : w_2)]$. The coordinate maps $\phi_k : U_k \to \mathbb{C}^2$ are defined using the two coordinates that are allowed to be zero. For example $\phi_0 : U_0 \to \mathbb{C}^2$ is defined by $\phi_0([(1 : Z_1 : Z_2)]) = (Z_1, Z_2)$. To explain the notation $\phi_0([(\cdots)])$, $[(1 : Z_1 : Z_2)]$ is the equivalence class of $(1 : Z_1 : Z_2)$, and, if $p = [(1 : Z_1 : Z_2)]$ then $\phi_0(p) = (z_1, z_2)$, where $z_1 = Z_1$ and $z_2 = Z_2$. The coordinate map for U_1 is $\phi_1([(Z_0 : 1 : Z_2)]) = (Z_0, Z_2)$.

   A set $A \subseteq \mathbf{CP}^2$ is open if $\phi_k(A \cap U_k) \subseteq \mathbb{C}^2$ is open in the usual sense. In particular the coordinate patches $U_k \subset \mathbf{CP}^2$ are open sets. A sequence p_n converges to p as $n \to \infty$ if every open set containing p contains all but a finite number of the points p_n.

   (a) Show that $p_n \to p$ in $\mathbf{CP}^2$ if and only if there is a unique k (depending on the sequence) and an N so that if $n > N$, then $p_n \in U_k$ and $\phi_k(p_n) \to \phi_k(p)$.

   (b) Show that the overlap maps are one to one and differentiable. More specifically, define $U_{01} = U_0 \cap U_1$ and $V_{01} \subset \mathbb{C}^2 = \phi_0(U_{01})$, and $V_{10} \subset \mathbb{C}^2 = \phi_1(U_{01})$. The overlap map ψ_{01} takes V_{01} to V_{10}. If $z \in V_{01} \subset \mathbb{C}^2$ then $w = \psi_{01}(z)$ means there is a $p \in U_{01}$ so that $\phi_0(p) = z$ and $\phi_1(p) = w$. *Differentiable* means that the four partial derivatives $\frac{\partial w_1}{\partial z_1}$, etc., exist in the usual sense of derivatives of analytic functions of a complex variable. *Hint.* This is harder understanding than to solve. Don't worry if your solution seems trivial.

   (c) Show that $\mathbf{CP}^2$ is compact in the sense that every sequence has a convergent subsequence.

(d) A set $D \subset \mathbf{CP}^2$ is compact if the three images $\phi_k(D \cap U_k) \subset \mathbb{C}^2$ are compact (take this as the definition). Show that D is compact if and only if any open cover has a finite sub-cover. You may use this property for $\mathbb{C}^2$.

(e) Show that $\mathbf{CP}^2$ is simply connected. *Warning.* Make sure your proof doesn't apply to $\mathbf{RP}^2$, which is not simply connected.

2. A "curve" in some two dimensional space is either the image of a function of one variable or the zero set of a non-degenerate function of two variables. The implicit function theorem says that these pictures are *locally* equivalent. A *projective* curve is a curve in projective space such as $\mathbf{CP}^2$. A projective *algebraic* curve wants to be the zero set of a non-degenerate polynomial

$$X = \{[Z] \text{ with } F(Z) = 0\} \ . \tag{1}$$

This definition has the problem that if $W \sim Z$ (i.e., $W = aZ$ in $\mathbb{C}^3$ for some $a \in \mathbb{C}$) it might be that $F(Z) = 0$ but $F(W) \neq 0$. This is fixed by requiring F to be a *homogeneous* polynomial, which means that every monomial in F has the same degree. In more detail, a *monomial* is a product of variables such as Z_0 (degree 1), or $Z_0 Z_1 Z_2^2$ (degree 4). A homogenous polynomial is a linear combination (with complex coefficients) of monomials of the same degree. Equivalently,

(a) Show that F is a linear combination of monomials of degree d if and only if $F(aZ) = a^d F(Z)$ for all $Z \in \mathbb{C}^3$.

(b) Show that if F is homogeneous of degree d then

$$F(Z) = d \cdot Z \cdot \nabla F(Z) = \sum_{j=0}^{3} Z_j \frac{\partial F(Z)}{\partial Z_j} \ .$$

This is (alas!) called "Euler's homogeneous function theorem". Euler got around.

(c) Show that if F is homogeneous of degree $d \geq 1$ then (1) defines a subset $X \subset \mathbf{CP}^2$. Such as set is called a *projective algebraic curve.* Show that such an X is compact.

(d) An *affine algebraic curve* is defined as the zero set of a polynomial in two complex variables, such as $z_2^2 = z_1^3 - az_1 - b$ (more properly, $f(z_1, z_2) = z_1^3 - az_1 - b - z_2^2 = 0$). Show that any affine algebraic curve is the intersection of a projective algebraic curve with the coordinate patch U_0 from Exercise 1.

(e) Suppose the affine curve is defined by $z_2^2 = z_1^3 - z_1 - 1$. Show that this is the intersection with U_0 of a projective curve with a polynomial $F(Z)$ of homogeneous degree 3. Write such an F. Show that the projective curve is a Riemann surface. This involves using the implicit function theorem (which we used earlier for affine curves) at every point of X. Some of these are not in U_0, but then they are in U_1 or U_2.

3. The Weierstrass P-function is (the Latex command is `\wp(z)`, where "wp" is for Weierstrass P fnction)

$$\wp(z) = \frac{1}{z^2} + \sum\nolimits'_{nm} \left[\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2} \right] . \tag{2}$$

The related power sums are called *Eisenstein series*:

$$G_{2k} = \sum\nolimits'_{nm} \frac{1}{(n+m\tau)^{2k}} . \tag{3}$$

You may assume that $\wp(z+1) = \wp(z)$ and $\wp(z+\tau) = \wp(z)$. This exercise gives an approach to the Taylor series of $\wp$ that illustrates some basic complex analysis tricks.

(a) Suppose f is a sum of analytic terms

$$f(z) = \sum_k a_k(z)$$

and that the sum converges locally uniformly in the sense that for any R

$$\sum_k |a_k(z)| \leq C(R) \; , \quad \text{if } |z| \leq R \; .$$

Define the definite integrals starting at $z = 0$ along any path to z

$$g(z) = \int_0^z f(\zeta) d\zeta \; , \quad b_k(z) = \int_0^z a_k(\zeta) d\zeta \; .$$

Show that the functions g and b_k are well defined in the sense that they are independent of the contour from 0 to z (easy). Show that

$$g(z) = \sum_k b_k(z) \; .$$

Show that the sum converges locally absolutely. *Conclusion.* You can do definite integrals of convergent series term by term.

(b) Let $a_{nm}(z)$ be the nm term in the $\wp$ function sum (2). Find a formula for

$$b_{nm}(z) = \int_0^z a_{nm}(\zeta)\, d\zeta \; .$$

(c) Define the $\mathcal{Q}$ function (this is not a standard term, the Latex command is `\mathcal{Q}(z)`)

$$\mathcal{Q}(z) = -\frac{1}{z} - \sum_{nm}{}' \left[\frac{1}{z-n-m\tau} + \frac{1}{n+m\tau} + \frac{z}{(n+m\tau)^2} \right] \tag{4}$$

Verify by direct calculation without using parts (a) or (b) that the sum converges locally absolutely and satisfies $\mathcal{Q}'(z) = \wp(z)$.

(d) Show that $\mathcal{Q}$ has the Laurent Taylor expansion

$$\mathcal{Q}(z) = -\frac{1}{z} + \sum_{k=1}^{\infty} G_{2k} z^{2k+1} .$$

Find the radius of convergence of this series.

(e) Show that $\mathcal{Q}$ is not a doubly periodic function even though its derivative is doubly periodic and its poles form a doubly periodic array (the *period lattice*. We showed that $\wp$ is doubly periodic using the fact that $\wp'$ is doubly periodic, but that argument does not apply here. Why not?

(f) Find the Laurent Taylor expansion of $\wp$ about $z = 0$ in terms of the Eisenstein sums G_{2k}.

4. The Weierstrass P-function is (the Latex command is `\wp(z)`, where "wp" is for Weierstrass P fnction)

$$\wp(z) = \frac{1}{z^2} + \sum\nolimits_{nm}' \left[\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2} \right] .$$

The related power sums are called *Eisenstein series*:

$$G_{2k} = \sum\nolimits_{nm}' \frac{1}{(n+m\tau)^{2k}} . \tag{5}$$

You may assume that $\wp(z+1) = \wp(z)$ and $\wp(z+\tau) = \wp(z)$.

(a) Show that if $f(z)$ is an analytic function in a neighborhood of zero with $f(z) = a_2 z^2 + a_4 z^4 + O(|z|^6)$, then $f'(z) = 2a_2 z + 4a_4 z^3 + O(|z|^5)$. Give an example of a real smooth function $g(x)$ defined for real x that does not have this property.

(b) Show that $\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + a_6 z^6 + O(z^8)$ and find formulas for a_k in terms of s_k.

(c) Show that

$$\left(\wp'(z)\right)^2 - 4\wp(z)^3 = -60 G_4 \frac{1}{z^2} - 140 G_6 + O(z^2) .$$

We got the 60 in class but not the 140.

5. The *gamma function* is

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} \, dt . \tag{6}$$

We saw that $\Gamma(n+1) = n!$ and Stirling's approximation

$$\Gamma(s+1) = \sqrt{2\pi s}\, s^s e^{-s} \left[1 + O\left(\tfrac{1}{s}\right) \right] , \quad \text{as } s \to \infty . \tag{7}$$

The proof of Sterling we gave may have assumed s is real, but it applies also when s is replaced with $s + i\tau$ and τ is bounded as $s \to \infty$. If s is not real, s^s is interpreted as $e^{s\log(s)}$ with the branch of log closest to the "real" one (the real log on the positive real axis). The first few parts are warm-up. The hard part is finding the convergence factors $e^{-\frac{s}{n}}$. The eventual formula involves the Euler γ constant

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right] \approx .58 \ . \tag{8}$$

(a) Show that the integral converges to an analytic function of s if $\mathrm{Re}(s) > 0$.

(b) Show that the limit (8) exists and is positive. Show that

$$\sum_{k=1}^{n} \frac{1}{k} = \log(n) + \gamma + O\big(\tfrac{1}{n}\big) \ . \tag{9}$$

Hint. The technical bound (9) and the basic limit (8) follow from

$$\frac{1}{k} \ - \ \int_k^{k+1} \frac{1}{x} dx \ = \ O\big(\tfrac{1}{k^2}\big) \ .$$

(c) Show directly that there is an analytic function $f(s)$ defined for s in a complex neighborhood of 0 so that $\Gamma(s) = \frac{1}{s} + f(s)$ for $s > 0$. [This is a complicated way of saying Γ has a simple pole with residue 1 at $s = 0$.]

(d) Show that for any positive integer n and $\mathrm{Re}(s) > 1$

$$\Gamma(s) = \frac{1}{s} \left[\prod_{k=1}^{n} \frac{1}{1+\frac{s}{n}} \right] \frac{\Gamma(s+n+1)}{n!} \ .$$

(e) For any fixed s show that

$$\frac{\Gamma(s+n+1)}{n!} = n^s \left[\ 1 + O\left(\tfrac{1}{n}\right) \ \right]$$

(f) Show that

$$n = \left(\prod_{k=1}^{n} e^{\frac{1}{k}} \right) e^{-\gamma} \left[\ 1 + O\big(\tfrac{1}{n}\big) \right]$$

(g) Put these facts together and take the $n \to \infty$ limit to derive the Euler product for the gamma function

$$\Gamma(s) = \frac{e^{-s}}{s} \prod_{k=1}^{\infty} \frac{e^{\frac{s}{k}}}{1+\frac{s}{k}} \ . \tag{10}$$

Show that the product converges (locally uniformly) absolutely for any $s \notin \mathbb{Z}$ and defines a meromorphic function in the whole complex plane. Said with different emphasis, the infinite product (10) defines a meromorphic function that is an analytic continuation of the integral (6) from the right half plane to the whole plane.

(h) Derive the weird Euler formula (there are many weird Euler formulas)

$$s\Gamma(s)\Gamma(-s) = \frac{\pi}{\sin(\pi s)} \ .$$

You may use the product formula for $\sin(\pi s)$. *Remark.* This formula relates factorials (the gamma function) to trig functions. Euler also discovered a formula relating exponentials to trig functions: $e^{ix} = \cos(x) + i\sin(x)$. Both formulas require extending some function (factorial, exponential) beyond its original range of definition.