Complex Variables II
Assignment 2

1. (*I gave an overly complicated proof of a basic normal families lemma in class. Here is an approach I like better.*)

   Definition. Let $\mathcal{F}$ be a family of analytic functions defined on a domain (connected open set) Ω. The family is *uniformly bounded on compacts* if, for every compact subset $K \subset \Omega$ there is an M_K so that

   $$|f(z)| \leq M_K \ \text{ for all } z \in K \ \text{ and all } f \in \mathcal{F} \ .$$

   Definition. The *derivative family* of $\mathcal{F}$ is the set of derivatives of functions in $\mathcal{F}$:

   $$\mathcal{F}' = \{ \, f' \mid f \in \mathcal{F} \} \ .$$

   Lemma. If $\mathcal{F}$ is uniformly bounded on compacts, then $\mathcal{F}'$ is uniformly bounded on compacts.

   This exercise gives one path to the proof of the lemma. It assumes some facts about compact sets and continuous functions that should be familiar from Analysis. Please comment on the class discussion site if some of it is not familiar.

   (a) For $z \in K$, define its distance to the complement of Ω by

   $$R(z) = \text{dist}(x, \Omega^c) = \min_{\zeta \notin \Omega} |\zeta - z|$$

   Show that the minimum defining R is "achieved" (there is a $\zeta \notin \Omega$ with $R(z) = |\zeta - z| = \inf \cdots$) and that $R(z)$ is a continuous function of z. For continuity, it may be easier to show the concrete inequality $|R(z_1) - R(z_2)| \leq |z_1 - z_2|$. Be aware that Ω^c is closed but may not be compact.

   (b) Define the distance from K to Ω^c to be

   $$R_K = \text{dist}(K, \Omega^c) = \min_{z \in K} R(z) \ .$$

   (This is the *Hausdorff distance*.) Show that the minimum is achieved and that $R_K > 0$.

   (c) Define K' with $K \subset K'$ to be the set of points within $\frac{1}{2} R_k$ of K:

   $$K' = \left\{ \zeta \text{ so that } \text{dist}(\zeta, K) \leq \frac{1}{2} R_K \right\} \ .$$

   Show that K' is compact and that $K' \subset \Omega$.

(d) Show that if $z \in K$, then

$$|f'(z)| \leq \frac{2}{M_{K'}} R_K \ . \tag{1}$$

(You may use the argument from class for this.) Explain how this proves the lemma.

(e) Show that a family that is uniformly bounded on compacts is *locally Lipschitz*. More precisely, show that if $z \in \Omega$, then there is an $R > 0$ and an $L < \infty$ so that if $|z_1 - z| \leq R$ and $|z_2 - z| \leq R$, and if $f \in \mathcal{F}$, then

$$|f(z_1) - f(z_2)| \leq L\,|z_1 - z_2| \ .$$

2. Suppose $\mathcal{F}$ is locally bounded on compacts. Suppose that $f_n \in \mathcal{F}$ for each n and that $f_n(z) \to f(z)$ for each $z \in \Omega$. Do not assume that the convergence is uniform over z, which turns out to be a consequence. Do not assume $f \in \mathcal{F}$. Show that f is analytic in Ω and that $f_n'(z) \to f'(z)$ for all $z \in \Omega$.

3. (*This is an easier approach to Exercise 6 from Assignment 1.*) Consider the family of functions, defined for $0 < n \leq \infty$,

$$f_n(z) = \int_{-n}^{n} e^{-\frac{1}{2}(t-z)^2}\, dt \ .$$

Take $f_\infty(z)$ to be the $f(z)$ from Assignment 1, Exercise 6. Show that this family, excluding f_∞ is locally bounded on compacts. Show that $f_n(z) \to f_\infty(z) = f(z)$ as $n \to \infty$ for every $z \in \mathbb{C}$. Use this to show that $f(z)$ is analytic and is equal to the formal derivative:

$$f'(z) = \int_{-\infty}^{\infty} (t-z)e^{-\frac{1}{2}(t-z)^2}\, dt \ .$$

Use this formula to give another proof that $f'(z) = 0$.

4. The Riemann zeta function sum is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ . \tag{2}$$

We use the notation $s = \sigma + it$ (real and imaginary parts) and define n^s to be the analytic function $n^s = e^{s\log(n)}$, using the real positive log of the real positive integer n. Here are some basic facts about ζ.

(a) Show that the sum converges absolutely if $\sigma > 1$. *Hint.* Consider (and prove if you use) the inequality

$$\left|\frac{1}{n^s}\right| \leq \int_{n-1}^{n} \frac{1}{x^\sigma} dx \ .$$

(b) Let Ω be the open right half plane $\Omega = \{\,\mathrm{Re}(s) > 1\}$. Show that the partial ζ sums are uniformly bounded on compacts in Ω:

$$\zeta_k(s) = \sum_{n=1}^{k} \frac{1}{n^s} .$$

Conclude that $\zeta(s)$ is analytic in Ω with

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^s} .$$

(c) Show that if $\sigma > 0$, then there is a C so that

$$\left| \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx \right| \leq \frac{C}{n^{\sigma+1}} .$$

(d) Show that the following family is locally bounded on compacts in the larger half plane $\Omega' = \{\,\mathrm{Re}(s) > 0\,\}$:

$$f_k(s) = \zeta_k(s) - \int_1^{k+1} \frac{1}{x^s} dx .$$

(e) Show that there is a meromorphic function $\zeta(s)$ defined for $\mathrm{Re}(s) > 0$ that is equal to the zeta sum (2) for $\mathrm{Re}(s) > 1$ (this is called an *analytic extension*) so that the analytic extension is analytic except for a simple pole at $s = 1$ with residue 1. Another way to say this is that $\zeta(s) - \frac{1}{s-1}$ is analytic in $\mathrm{Re}(s) > 0$.

5. It can be easier to calculate derivatives by power series calculation than using implicit differentiation and the chain rule.

(a) (*General theory*) Suppose $f(z)$ is analytic in a neighborhood of $z = 0$ with a double root at $z = 0$, which means that $f(0) = f'(0) = 0$, $f''(0) \neq 0$. Show that there are two analytic functions $g_\pm(z)$ defined in a neighborhood of $z = 0$ so satisfy $g(z) = \sqrt{f(z)}$ in the sense that $g(z)^2 = f(z)$. *Hint* Factor z^2 out of the Taylor expansion of f.

(b) (*Application*) Consider the analytic function $g(z) = \sqrt{1 - \cos(z)}$ that is positive when z is real. Calculate $g^{(3)}(0)$ and $g^{(5)}(0)$ (the third and fifth derivatives). *Hint.*

$$(1+w)^{\frac{1}{2}} = 1 + \frac{1}{2}w - \frac{1}{8}w^2 + \cdots \quad \text{(why?)}$$

$$g(z) = \frac{1}{\sqrt{2}} z \left(1 - \frac{1}{12}z^2 + (??)z^4 + \cdots\right)^{\frac{1}{2}} .$$