Complex Variables II
Assignment 3

1. (*This is a version of the Laplace method discussed in class last Tuesday. A more general version may be easier to understand.*)

   The *Laplace method* (one of the tricks called "Laplace method") is a way to approximate certain integrals of the form
$$I(n) = \int_a^b e^{n\phi(x)} dx \; .$$
   The Laplace method finds an approximate value
$$I(n) \approx \sqrt{\frac{2\pi}{n\phi''_*}}\, e^{n\phi_*}$$
   The approximation depends on x_* being a maximizer of ϕ, with $\phi_* = \phi(x_*)$ and $\phi''_* = \phi''(x)$. The approximation applies as long as the integration interval contains x_*. Values $-\infty$ and/or $b = \infty$ are allowed. This Exercise verifies this under the hypotheses

   - (*Analyticity*) $\phi(x)$ is a real analytic function of x for $x \in [a, b]$. This is equivalent to $\phi(z)$ being complex analytic (complex differentiable) in an open set Ω that contains the real interval $[a, b]$.
   - (*Convexity*) $\phi''(x) < 0$ for all $x \in [a, b]$.
   - (*Local max*) There is an $x_* \in (a, b)$ with $\phi'(x_*) = 0$. To be clear, $a < x_* < b$ must be strict inequalities.

   The proof breaks the interval $[a, b]$ into three pieces with endpoints $a \leq x_1 < x_* < x_2 \leq b$, where the only the middle inequalities need to be strict.
$$\begin{aligned} I(n) &= J_1(n) + J_2(n) + J_3(n) \\ J_1(n) &= \int_a^{x_1} e^{n\phi(x)} dx \\ J_2(n) &= \int_{x_1}^{x_2} e^{n\phi(x)} dx \\ J_3(n) &= \int_{x_2}^{b} e^{n\phi(x)} dx \end{aligned}$$
   Define values $\phi_1 = \phi(x_1)$, $\phi'_1 = \phi'(x_1)$, etc. The proof consists of localizing to a small neighborhood of x_*, with an error bound for the parts left out, then approximating the central part using a quadratic approximation of ϕ.

(a) (*localization error bound*) Show that $\phi_1 < \phi_*$ and that

$$J_1(n) \leq \frac{1}{n\phi_1'} e^{n\phi_1} \; .$$

Show that $\phi_1 < \phi_*$ and $\phi_1') > 0$. Find a similar bound for J_3.

(b) (*Morse lemma for the central part*) Show that if $|x_1 \leq x_*|$ and $|x_* \leq x_2|$ are small enough, then there is an analytic function $y(x)$ defined for $x_1 \leq x \leq x_2$ with $y(x_*) = 0$ and[1] $y'(x_*) = 1$, $y'(x) > 0$ for $x_1 \leq x \leq x_2$, and

$$\phi(x) = \phi_* + \frac{1}{2}\phi_*'' y^2 \; .$$

This replaces the approximation $\phi(x) \approx \phi_* + \frac{1}{2}\phi_*'' (x - x_*)^2$ with an exact relation of the same form. *Hint.* This is almost identical to an exercise from Assignment 2. You can solve $\sqrt{\phi(x) - \phi_*} = Cy(x)$ by factoring out $(x - x_*)^2$ from the Taylor series of $\phi(x) - \phi_*$ about x_*. A trick says that if $f(x_*) \neq 0$ then there is an analytic $\sqrt{f(x)}$ defined near x_*. Make sure to check that the $y(x)$ is real for real x, which you might worry about given that we're using complex analysis to find it.

(c) Write the middle integral as

$$J_2(n) = C_1(n) \int_{y_1}^{y_2} e^{-nC_2 y^2} \frac{dx}{dy}(y)\, dy \; .$$

Use $\frac{dx}{dy} = 1 + C_3 y + O(y^2)$ and show that J_2 is "accurately" approximated by replacing $\frac{dx}{dy}$ by its two term Taylor approximation then taking $y_1 = -\infty$ and $y_2 = \infty$.

(d) Assemble these pieces to prove an inequality of the form

$$\left| I(n) - \sqrt{\frac{2\pi}{n\phi_*''}}\, e^{n\phi_*} \right| \leq \frac{C}{n^{\frac{3}{2}}} I(n) \; .$$

Note that just one of the error terms is as large as the right side of this inequality. Most are exponentially smaller.

(e) Apply these ideas to Stirling's approximation

$$n! = \int_0^\infty t^n e^{-t} dt = n^n e^{-n} \sqrt{2\pi n} \left(1 + O(\frac{1}{n}) \right) \; .$$

You can convert to the form of $I(n)$ as we did in class. Be aware that the ϕ you get is not analytic at $t = 0$. What can you do about that?

[1] This makes $y(x)$ a local *near identity transformation*, because $y \approx x - x^*$ near x_*.

2. The *Fresnel integral* with real a is

$$I(a) = \int_{-\infty}^{\infty} e^{ia\frac{x^2}{2}}\, dx \; .$$

Consider the half integral that goes from 0 to ∞. Suppose $a > 0$ and consider contours

$$\begin{aligned}
\gamma_0(R) &= [0, R]\;, \quad \text{(the real interval)} \\
\gamma_1(R) &= [0, (1+i)R] = \{(1+i)t \text{ with } 0 \leq t \leq R\} \\
\gamma_2(R) &= [R, (1+i)R] = \{(R+i)t \text{ with } 0 \leq t \leq R\}
\end{aligned}$$

Show that

$$\int_0^R e^{ia\frac{x^2}{2}}\, dx = \int_{\gamma_1(R)} e^{ia\frac{z^2}{2}}\, dz - \int_{\gamma_2(R)} e^{ia\frac{z^2}{2}}\, dz$$

Show that

$$\begin{aligned}
&\int_{\gamma_1(R)} e^{ia\frac{z^2}{2}}\, dz \to \text{ a finite number as } R \to \infty \\
&\int_{\gamma_2(R)} e^{ia\frac{z^2}{2}}\, dz \to 0 \text{ as } R \to \infty \; .
\end{aligned}$$

Use these ideas to show that the Fresnel integral with real a converges (though not absolutely) and find its value for $a > 0$.

3. The *Bessel function* of order n and argument r is defined by the integral

$$J_n(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\cos(\theta) - in\theta} d\theta \; .$$

(*Warning.* Other definitions of $J_n(r)$ may differ from this in having sin instead of cos and/or + instead of −. These differences don't change the problem.) Show that $J_n(r)$ goes to zero exponentially as $n \to \infty$ for any fixed r. For this Exercise, a quantity Q is exponentially small if there is are positive C_1 and C_2 so that $|Q| \leq C_1 e^{-C_2 n}$. *Hint.* Interpret this as a contour integral (there are several ways to do that) and move the contour to make the integrand exponentially small. You are not being asked to find the actual behavior of $J_n(r)$ as $n \to \infty$, though that is possible. You are just being asked to show it is exponentially small.