Complex Variables II
Assignment 5

1. This exercise gives a derivation of the Poisson kernel formula for a harmonic function based on its boundary values. In this version, if $\phi(x, y)$ is bounded and harmonic in a neighborhood of the right half plane ($y \geq 0$) then
$$\phi(1,0) = \int_{-\infty}^{\infty} P(y,1)\,\phi(y,0)\,dy\;. \tag{1}$$

   (a) Show that if u is harmonic in a neighborhood of the closed unit disk then
$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta \tag{2}$$
   Show that this is a consequence of the Cauchy formula for analytic functions
$$f(0) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z}dz\;.$$
   [Please do not use the PDE derivation, which uses Green's theorem and/or the divergence theorem. This is *complex analysis*.]

   (b) Show that the mapping $w = \frac{1+z}{1-z}$ gives a conformal equivalence between the unit disk and the right half plane, except that $z = 1$ is mapped to $w = \infty$.

   (c) Show that if ϕ is bounded and harmonic in a neighborhood of the right half plane, and if $u(z) = \phi(z(w))$, then the formula (2) holds for u. Warning: you may have to think about whether $\phi(z(w))$ satisfies enough conditions for (2) to hold.

   (d) Derive the right half plane formula (1) from the circle formula (2) by changing variables, thinking of $z = e^{i\theta}$ as a function of $w = iy$.

   (e) Show that the Poisson formula (1) generalizes to a formula for $u(x, 0)$ for any $x > 0$

   (f) Let w be an arbitrary point in the open right half plane. Find the conformal transformation of the right half plane to itself that takes $(1, 0)$ to w. *Hint.* This is easy.

   (g) Use the result of part (e) to find a formula for $\phi(w)$ in terms of its boundary values $\phi(0, y)$ and a kernel function $P(y_0, x, y)$ so that
$$\phi(x,y) = \int_{-\infty}^{\infty} P(y_0,x,y)\,\phi(y_0)\,dy_0\;.$$

(h) Use the inverse of the map from part (b) to show that there is an integral formula for the values of a harmonic function u defined for z inside the open disk:

$$u(z) = \frac{1}{2\pi} \int \widetilde{P}(z,\theta)\, u(e^{i\theta}\, d\theta \ .$$

It is not necessary to find a formula for $\widetilde{P}$, only to see that it is differentiable as a function of z. This allows you to see that

(i) Show that if u is harmonic in the open unit disk and continuous on the closed unit disk, then u is differentiable in the open disk and

$$|\nabla u(z)| \leq C(z) \max_{\theta} \left| u(e^{i\theta} \right| d\theta \ .$$

2. An *oscillatory integral* is an integral whose value is small because of cancellation in the integrand. The Fresnel integral is an example. The integrand has absolute value 1 over the range $[-\infty, \infty]$. The integral of the absolute value is infinite, but the integral itself is finite, because of cancellations.

$$F = \int_{-\infty}^{\infty} e^{ix^2} dx = \frac{1+i}{2} \sqrt{\pi} \ , \quad \int_{\infty}^{\infty} \left| e^{ix^2} \right| dx = \infty$$

The intuition of cancellation may be more clear in the real part:

$$F = \int_{-\infty}^{\infty} \cos(x^2)\, dx \ . \tag{3}$$

The integral of $\cos(t)$ is zero over a period from $2\pi n$ to $2\pi(n+1)$. The integrand $\cos(x^2)$ goes through one period when x goes from are $x_n = \sqrt{2\pi}\sqrt{n}$, to x_{n+1}. The integral over this interval is not exactly zero because x^2 is not a linear function of x,

$$\int_{x_n}^{x_{n+1}} \cos(x^2)\, dx \neq 0 \ .$$

However, some "analytical technique" shows that

$$\int_{x_n}^{x_{n+1}} \cos(x^2)\, dx = O(|n|^{-\frac{3}{2}}) \tag{4}$$

This can be used to give a more elementary (no complex analysis) yet more complicated proof that the infinite domain integral (3) converges.

Assignment 3 gave an example of a real integral of the form

$$\int_a^b e^{r\phi(x)} dx$$

that whose value is determined (approximately) near x values with $\phi'(x) = 0$, assuming that $\phi'' \neq 0$ there. This can be true also of oscillatory integrals, though the phenomenon is more subtle to see and to prove. Consider an integral

$$A(r) = \int e^{ir\phi(x)} dx \ . \tag{5}$$

The function $\phi(x)$ is the *phase function*.[1] Places where $\phi'(x) = 0$ are *stationary phase* points. Approximating using stationary phase points is the *method of stationary phase*. The calculations of stationary phase seem similar to calculations of the Laplace approximation of Assignment 3, but the integrands look different, oscillatory here, sharply peaked there).

It may be possible, if the phase function is real analytic, to use the *saddle point method* (also called the *method of steepest descent*, which is not to be confused with gradient descent optimization). Suppose x_* is a stationary phase point (i.e., $\phi'(x_*) = 0$) and $\phi''(x_*) \neq 0$. Then the simplest non-constant Taylor approximation of ϕ near x_* is

$$\phi(x) \approx \phi(x_*) + \frac{1}{2}\phi''(x_*)(x - x_*)^2 \ .$$

If you use this approximation in the integral (5) and integrate the resulting Fresnel integral exactly, you get (writing ϕ_* for $\phi(x_*)$, etc.)

$$A(r) \approx e^{it\phi_*} \frac{\sqrt{2\pi}}{\pm\sqrt{ir\phi''_*}} \ . \tag{6}$$

This saddle point method gives a rigorous justification for this approximation in some cases.

In class we applied this to the Airy function. Here, we apply it to the Bessel function

$$J_0(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\cos(x)} dx \ . \tag{7}$$

(a) Verify the inequality (4).

(b) Identify the stationary phase points in the integral (7). For this, it is convenient to consider the contour γ_0 for (7) as being a real interval of length 2π that starts and ends somewhere besides 0 or π. *Hint* $x_+ = 0$, $x_- = \pi$.

(c) Show that the integral over γ_0 is equal to an integral over a contour with two pieces γ_+ and γ_-. These are defined by $\gamma_\pm(x) = (x, y_\pm(x))$, which means, for example, that γ_+ is defined "over" a segment of the real axis. The range for γ_+ is $-\frac{\pi}{2} < x < \frac{\pi}{2}$. The contour γ_+ is defined by $\mathrm{Re}(\phi(\gamma_+)) = \mathrm{Re}(\phi(x_+)) = 1$. Choose γ_+ so that $i\phi(\gamma_+(x))$ has a local max at $x = x_+ = 0$.

[1] The function $e^{i\theta}$ is a periodic function of θ, which means it repeats itself. The number θ tells you where you are in this repeating cycle. Therefore, θ might be called the "phase", thinking of phases of that the moon goes through each four weeks.

(d) Use the Laplace method from Assignment 3 to show that

$$\int_{\gamma_+} e^{ir\phi(z)} dz$$

has an approximation of the form (6).

(e) Explain how this needs to be modified to handle the stationary phase point x_π. You do not have to do all the details, but explain the similarities and differences between the two cases.