Complex Variables II
Assignment 6

1. This exercise gives another derivation of the Poisson kernel formula for the disk. We are given a function $f(\theta)$ that is a periodic function of θ with period 2π. We want a harmonic function $u(x,y)$ defined for $x^2+y^2<1$ that has the *boundary values* f. This means that $u(r\cos(\theta), r\sin(\theta)) \to f(\theta)$ as $r\to 1$ with $r<1$. The solution of this *boundary value problem* takes the form of an integral[1]

$$u(r\cos(\theta), r\sin(\theta)) = \int_0^{2\pi} P(r,\phi) f(\theta-\phi)d\phi \tag{1}$$
$$= \int_0^{2\pi} P(r,\theta-\phi) f(\phi)d\phi \; .$$

The Poisson kernel is an *integral representation* of the harmonic function u as a *convolution* of the Poisson kernel $P(r,\theta)$ with the *boundary data* f. The formula (1) may seem arbitrary, but it has motivations

- u is expressed as an integral involving f because u is a linear function of f. Suppose that there is a solution to the boundary value problem (a u for any given f), and that the solution is unique (there is only one harmonic u with boundary values f). Then the solution is a linear function of the data, which means that if u_1 has boundary values f_1 and u_2 has boundary values f_2, then u_1+u_2 has boundary values f_1+f_2, and cu_1 has boundary values cf_1 (c being a constant). If you fix a point (x,y) inside the disk, then the number $u(x,y)$ is a linear "functional" of the data function f.[2] This means that the values of f along the boundary of the disk should be combined in a linear way (i.e., integrated) to produce u.
- The boundary value problem is *rotation invariant*. Informally, if you rotate the problem by angle α, then the answer to the rotated problem is the answer to the original problem, rotated by α. More technically, if u is the harmonic function with boundary values f, and if $f_\alpha(\theta) = f(\theta+\alpha)$, then the corresponding harmonic function is $u_\alpha(r\cos(\theta, r\sin(\theta)) = u(r\cos(\theta+\alpha, r\sin(\theta+\alpha))$. This implies that if you have a formula for u when $\theta = 0$, then you get formulas for other θ by rotation. If

$$u(r,0) = \int_0^{2\pi} P(r,\phi) f(-\phi)d\phi \; ,$$

[1] The integrals are equal in part because f and P are periodic, so the range of integration may be any interval in ϕ of length 2π. Keep this equivalence in mind when doing this exercise.
[2] *Functional* means function of a function.

then (1) applies for other θ values. The minus sign $f(-\phi)$ makes the $\theta = 0$ formula more complicated, but it puts the general formula (1) into the form of a convolution.

The Poisson kernel function P may be found using Fourier series. We saw that "any" periodic function f may be expressed as a Fourier series

$$f(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta} \ , \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta) d\theta \ . \tag{2}$$

Therefore, if we can solve the boundary value problem for $f(\theta) = e^{in\theta}$, then we can add these solutions to get the solution for a general f. Warning: The original boundary value problem was for real boundary values f and real valued harmonic functions u. Now we allow u and f to have complex values. It will turn out (hopefully!) that you get a real u if the boundary values are real. Be careful not to think u should be complex analytic as a function of $z = x + iy$ just because it has complex values.

(a) Show that $u(x, y) = r^{|n|}e^{in\theta}$ solves the boundary value problem with boundary values $f(\theta) = e^{in\theta}$. *Hint.* This is "clear" for $n \geq 0$, but for $n < 0$ you need a trick. You might with with real and imaginary parts involving cos and sin or you might use the complex conformal mapping $z \to \frac{1}{z}$, or you might calculate directly.

(b) Combine this with the Fourier representation (2) to get

$$P(r, \theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} r^{|n|}e^{in\theta} = \frac{1}{2\pi}\sum_{1}^{\infty} r^{n}e^{in\theta} + \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{-\infty}^{-1} r^{-n}e^{in\theta} \ .$$

(c) Add and combine the geometric series from part (b) to get a simple explicit formula for $P(r, \theta)$.

2. (*Maximum principle and uniqueness for harmonic functions*) Assignment 5 included the *mean value* formula

$$u(x_0, y_0) = \frac{1}{2\pi r}\int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta \ .$$

This holds if u is harmonic in a domain that contains the disk of radius r about the point (x_0, y_0). Suppose that u is real and

$$M = \max_{0\leq\theta\leq 2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) \ .$$

(a) Show that $u(x_0, y_0) \leq M$ and that if $u(x_0, y_0) = M$ then $u(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) = M$ for all θ.

(b) Show that if u is harmonic in the open unit disk and continuous in the closed unit disk, then

$$\max_{x^2+y^2\leq 1} u(x, y) = \max_{x^2+y^2=1} u(x, y) \ .$$

(c) Suppose u and v are both harmonic in the open disk and continuous in the closed unit disk. Show that if $u(x,y) = v(x,y)$ with $x^2+y^2 = 1$, then $u(x,y) = v(x,y)$ for all (x,y) in the unit disk. Conclude that the Poisson integral formula (1) gives the unique solution to the boundary value problem if it defines a continuous function u.

3. Find a formula for the conformal map that takes the upper half disk $H = \{z \text{ with } |z| < 1\ ,\ \text{Im}(z) > 0\}$ to open unit disk. *Hint.* One way to do this: (1) throw 1 to ∞ while preserving the real axis and sending -1 to 0, (2) open the quadrant to a half plane, (3) map the half plane to the disk.

4. In Exercise 3, the bottom of the half disk, $B = \{z = x + iy \text{ with } x \in [-1,1]\ ,\ \ y = 0\}$ is mapped to an arc of the circle $A = \left\{w = e^{i\theta} \text{ with } \theta_0 \leq \theta \leq \theta_1\right\}$. What arcs are possible? *Hint.* This is mostly about conformal maps of the disk to itself.

5. The *Chebychev polynomials*[3] are

$$T_n(x) = \cos(n\theta)\ ,\quad x = \cos(\theta)\ .$$

That is, given $x \in (-1,1)$, first you find θ with $x = \cos(\theta)$, then you use that θ to define $T_n(x)$. They are used in many sophisticated numerical algorithms.

(a) Calculate T_0 and T_1 (for future parts).

(b) Show that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. *Hint.* $\cos(n\theta + \theta) = \cdots$, $\cos(n\theta - \theta) = \cdots$.

(c) Show that $T_n(x)$ is a polynomial of degree n with $T_n(x) = 2^n x^n + \cdots$.

(d) Show that x^n is *almost* a polynomial of degree $n-1$ in the sense that there is a polynomial $r_{n-1}(x)$ of degree $n-1$ so that $|x^n - r_n(x)| \leq 2^{-n}$ if $x \in [-1,1]$. *Hint.* $|T_n(x)| \leq 1$ for $x \in [-1,1]$, why?

(e) Suppose $f(x)$ is defined for $x \in [-1,1]$. Define $g(\theta) = f(x)$, with θ and x related as before. Show that if f is analytic in a complex neighborhood of $[-1,1]$ in the complex plane, then $g(z)$ is analytic in a complex neighborhood of the unit circle. *Hint.* Zhukowsky (sp?)

(f) Show that if f is real analytic as a function of the real variable x in $[-1,1]$, then

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)\ ,\quad \text{with } a_n = C_n \int_{-1}^{1} T_n(x) f(x) \frac{dx}{\sqrt{1-x^2}}\ .$$

Find C_n. Show that the sum converges geometrically, in that there is an M and an $r > 0$ with $r < 1$ so that $|a_n| \leq Mr^n$. The numbers C_n do not depend on f, but M and r depend on f. *Hint.* Use part (e).

[3] The name is written "Tchebycheff" in German because "ch" pronounced differently. Hence T_n instead of C_n.

These properties of Chebychev polynomials make them useful in numerical computing. Some properties are common to all families of *orthogonal polynomials*. The Chebychev polynomials satisfy the *orthogonality* relations

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{dx}{\sqrt{1-x^2}} = 0 \;, \quad \text{if } m \neq n \;.$$

A *weight function* is a non-negative $w(x)$ with $w(x) > 0$ for some open set of x and

$$\int_{-\infty}^{\infty} |x^n|\, w(x)\, dx < \infty \;, \quad \text{for all } n \;.$$

The Chebychev weight function is $w(x) = \left(1-x^2\right)^{-\frac{1}{2}}$ if $|x| \leq 1$ and $w(x) = 0$ otherwise. Orthogonal polynomials with respect to weight function w are polynomials $P_n(x)$ of degree n with

$$\int_{-\infty}^{\infty} P_n(x) P_m(x)\, dx = 0 \;, \quad \text{if } m \neq n \;.$$

Orthogonal polynomials with respect to the weight function $w(x) = 1$ for $|x| \leq 1$ and $w(x) = 0$ otherwise are *Legendre polynomials*. The weight function $w(x) = e^{-\frac{1}{2}x^2}$ defines *Hermite polynomials*. Each family of orthogonal polynomials satisfies a *three term recurrence relation* of the form $P_{n+1} = cxP_n - a_n P_n - b_n P_{n-1}$. Part (b) gives the specific three term recurrence relation for Chebychev polynomials. These are remarkable. Clearly $xP_n(x)$ has degree $n+1$ so general linear algebra tells us xP_n is a linear combination of the orthogonal polynomials P_{n+1}, P_n, $\cdots$, P_0. The surprising thing is that only P_{n+1}, P_n and P_{n-1} appear. A general function may be "expanded" in orthogonal polynomials as

$$f(x) = \sum_n a_n P_n(x) \;, \quad \text{with } a_n = C_n \int f(x) P_n(x) w(x)\, dx \;.$$

Part (f) verifies this for Chebychev polynomials.

Other facts are special to Chebychev polynomials. The relation between Chebychev expansions (part (f)) and Fourier series (part (e)) allows computational methods for Fourier series, such as the *Fast Fourier transform* (*FFT*) to be applied to Chebychev series. This makes Chebychev polynomials more useful than, say, Legendre polynomials for many practical tasks. The *Chebfun* computational package is built around Chebychev expansions. Part (d) illustrates the fact that the *monomial basis* $(1, x, x^2, \cdots)$ is not good for many practical computations. The powers are linearly independent in the sense of linear algebra, but they are "exponentially close" to being linearly dependent, as x^n is within 2^{-n} of the subspace spanned by $1, x, \cdots, x^{n-1}$.