Complex Variables II
Assignment 7

Corrections. Exercise 2 has been edited to make it more clear and correct typos.

For this assignment you may use the following theorems without giving proofs.
Riemann mapping theorem. Any simply connected open set $\Omega \subset \mathbb{C}$ with $\Omega \neq \mathbb{C}$ is conformally equivalent to the open unit disk.
Bounded convergence theorem (*not the most general version*) Let $f(x, y)$ be a continuous function of $x \in [a, b]$ for each $y \in (0, 1)$. Suppose $|f(x, y)| \leq M$ for all $x \in [a, b]$ and $y \in (0, 1)$. Suppose $f(x, y) \to 0$ as $y \to 0$ for each x. Then

$$\int_a^b f(x, y) dx \to 0 \ , \quad \text{as } y \to 0 \ .$$

Notes: $f(x, y)$ need not be defined for $y = 0$, or you can think of $f(x, 0) = 0$. The theorem refers to $f(x, y)$ only for $y > 0$. The notation $y \in (0, 1)$ (the open interval) reflects this. The upper limit $y < 1$ is arbitrary in the sense that any other positive upper limit would be equivalent. The traditional statement and proof of the bounded convergence theorem involve measure theory, but the statement here does not involve measure theory, as f is a continuous function of x so the integral may be taken to be the Riemann integral.

1. This exercise is a review of stuff from Calculus II, but done more carefully. Prove the following assertions:

   (a) Suppose $f(x, y)$ and $g(x, y)$ are continuously differentiable and

   $$\partial_y f(x, y) = \partial_x g(x, y) \ .$$

   Suppose this holds in an open convex set $\Omega \subseteq \mathbb{R}^2$. Then there is a $\phi(x, y)$, which is continuously differentiable, so that

   $$\partial_x \phi(x, y) = f(x, y) \ , \quad \partial_y \phi(x, y) = g(x, y) \ .$$

   This holds for all $(x, y) \in \Omega$.

   (b) Let $u(x, y)$ be twice continuously differentiable and harmonic in an open convex domain Ω. Then there is a continuously differentiable complex conjugate function $v(x, y)$, defined in Ω, so that

   $$\partial_x u(x, y) = \partial_y v(x, y) \ , \quad \partial_y u(x, y) = -\partial_x v(x, y) \ .$$

   (c) The functions ϕ and v defined above are unique up to a constant.

(d) Show that if u is twice continuously differentiable in a neighborhood of a point, then u is harmonic if and only if u is the real part of a complex differentiable function defined in a neighborhood of that point. Conclude that a twice differentiable harmonic function is real analytic in the sense that the two variable Taylor series identity holds in a neighborhood of (x_0, y_0).

$$u(x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{j!k!}\partial_x^j \partial_y^k u(x_0,y_0)(x-x_0)^j (y-y_0)^k \ .$$

Hint. It may be easier to derive this from the complex Taylor series of $f = u + iv$ and the Cauchy Riemann equations than to do it directly.

2. Harmonic functions are an approach to boundary regularity of conformal maps. The technical core of this is analytic extension of harmonic functions defined in a half space. Let D be the open upper half disk $D = \big\{(x,y) \text{ with } y > 0,\ x^2 + y^2 < 1\big\}$. The interval $B = (-1, 1)$ is the bottom boundary of D, more correctly, $B = \{(x,y) \text{ with } -1 < x < 1,\ y = 0\}$. Suppose u is harmonic in D, bounded (there is M with $|u(x,y)| \leq M$ for all $(x,y) \in D$). This implies that u is real analytic in D, which is equivalent to the existence of an analytic function defined in D so that $u = \mathrm{Re}(f)$.

   In general, a bounded harmonic function can "act up" at the boundary. For example, the gradient of a bounded harmonic function does not have to be bounded. However, this changes if $u = 0$ on the boundary, because there is a *harmonic extension* of u across the boundary. This means that there is a function, also called u, that is equal to u in D but is defined in a full neighborhood of B. This puts B in the interior of the region where u is defined. Since the extension turns out to be bounded, we learn that u is in fact analytic in a full neighborhood of B. This implies that the harmonic conjugate is also analytic, so u is the real part of a function that is analytic in a neighborhood of B. This fact is called *boundary regularity*, because u is shown to be "regular" (analytic in this case) "up to the boundary" (in fact, even across the boundary).

   The analytic extension is defined by *reflection*, which is $u(x,-y) = -u(x,y)$, for a suitable range of x and y. This anti-symmetric extension implies that $u = 0$ if $y = 0$ (why?). It turns out that, conversely, if $u = 0$ on B, then the reflection defines a function for some range of $y < 0$, which is the harmonic extension. The textbook has a discussion of harmonic and analytic continuation, but it isn't complete. The book by Alfors has an elegant and complete discussion. The present exercise is loosely based on the discussion of Alfors. The ultimate goal is to show that conformal mappings defined by the Riemann mapping theorem are regular up to the boundary. The proof of the Riemann mapping theorem makes the conformal mapping bounded if the image of the mapping is a bounded set such as a disk. However, it does not imply that the conformal mapping is continuous on the closure of the region being mapped. Indeed, derivatives of the mapping must "blow up" at the boundary if the boundary is a bad set like the graph of $\sin(1/x)$.

We suppose (for reasons that will be discussed in class and are made clear in Alfors) that u is bounded on D but we do not assume u is continuous, or even defined, on $D \cup B$ (i.e., "up to the boundary"). Instead, we suppose that $u = 0$ on B in the sense that $u(x, y) \to 0$ for any x as $y \downarrow 0$. This exercise shows that $u = 0$ on B in this weak sense is enough to define a harmonic extension that is analytic in a neighborhood of B. The harmonic extension of u up to and across the whole boundary is given by the *reflection principle* (there is more than one "principle" with this name) $u(x, -y) = -u(x, y)$ for all $(x, y) \in D \cup B$. The proof is a little technical because we start not knowing much about u on B.

(a) Let $D_{r,h} \subseteq D$ be the "radius" r half disk raised by h above the $x-$axis. That means that $D_{r,h}$ is the set elements of D with $y > h$ and $x^2 + (y - h)^2 < r^2$. The simple contour on the boundary of $D_{r,h}$ has the semicircular part $\lambda_{r,h}(t) = ih + re^{it}$, $0 \leq t \leq \pi$ and the bottom part $\beta_{r,h}(t) = ih + t$, $-r \leq t \leq r$. Show that there is a "Poisson kernel" $P_{r,h}(x, y, \xi, \eta)$ so that

$$u(x,y) = \int_{\beta_{r,h}} P(x, y, \beta_{r,h}(t))u(\beta_{r,h}(t))dt + \int_{\lambda_{r,h}} P(x, y, \lambda_{r,h}(t))u(\lambda_{r,h}(t))dt \ .$$

The first integral on the right may be written as

$$\int_{-r}^{r} P(x, y, t, h)u(t, h)dt \ .$$

The second integral on the right also has an explicit expression in terms of the euclidian coordinates of $\lambda_{r,,h}(t)$. *Hint.* We have seen that conformal mappings map Poisson kernel representations on one domain to Poisson kernel representations on another domain. If the conformal map is given explicitly, you know its behavior on the boundary. It is not necessary to find a formula for P, only to understand its qualitative properties.

(b) Show that if $(x, y) \in D_{r,0}$, and if $u = 0$ on B in the sense described above, then you can take the $h \to 0$ limit of the integral representations from part (a) to get

$$u(x,y) = \int_{\lambda_{r,0}} P(x, y, re^{it})u(re^{it})dt \ .$$

(c) Show that $P(x, y, z)$ is a harmonic function of (x, y) for any $z \neq (x, y)$. *Hint.* The property of being harmonic is preserved under conformal transformations.

(d) Show that the u represented by the integral of part (b) is analytic in a neighborhood of a smaller interval on the $x-$axis, including an open part of the lower half plane.

(e) Let $f(z)$ be an analytic function with $u(x,y) = \mathrm{Re}(f(x+iy))$ for $(x,y) \in D$. Show that $f(x+iy)$ has an analytic extension to a complex neighborhood of the origin, and that this f satisfies the *reflection principle* $f(x-iy) = \overline{f}(x+iy)$ in a neighborhood of the origin. This shows that f has an analytic continuation to some open part of the lower half plane.

3. Algebraic functions are functions that arise by solving polynomial equations whose coefficients are polynomials in z. An example is

$$f(z) = \sqrt{(z-1)(z+1)} = \sqrt{z^2-1}\,. \tag{1}$$

The points $z = \pm 1$ are *branch points*. You can think of this informally as the solution to the equation $w^2 = z^2 - 1$. A more generic example would be

$$w^3 + zw^2 + (z+4-1)w + z^7 + z^5 + 1 = 0\,.$$

More famous examples are *elliptic curves* defined by equations quadratic in w and cubic in z such as

$$w^2 = z^3 + az + b\,.$$

This exercise starts the exploration of the quadratic case (1). The *slit plane* is the complex plane with the interval (the "slit") $[-1,1]$ removed. The slit is a contour that connects the two branch points.

(a) Suppose (w_0, z_0) is a solution to $w^2 = z^2 - 1$. Assume z_0 is not one of the branch points. Show that there is an analytic function $w = f(z)$ defined in a neighborhood of z_0 so that $w(z)^2 = z^2 + 1$. You may do this explicitly or with a more generic argument.

(b) Show that there are exactly two functions $f_\pm(z)$ defined in the slit plane so that $f_\pm(z)^2 = z^2 - 1$.

(c) Show that there is no analytic function $f(z)$ that satisfies $f(z)^2 = z^2 - 1$ defined in the complex plane minus the branch points. *Hint.* Consider a small circle about the branch point $z = 1$, $\gamma(t) = a + re^{it}$, with r small. Show that if $f(\gamma(0)) = f_+(\gamma(0))$, and if $\frac{d}{dt}f(\gamma(t))$ is what calculus says it should be, then $f(\gamma(2\pi)) = f_1(\gamma(0))$. We say that this contour goes from one "sheet" of the Riemann surface defined by (1) to the other sheet.

4. (*Partitions, and the generating function here, have led to surprising and deep mathematics, including amazing theorems of Hardy and Ramanujan.*)

A *partition* of a positive integer n is a representation of n as a sum of positive integers. For example, the partitions of 4 are

$$\begin{aligned} 4 &= 1+1+1+1 \\ 4 &= 1+1+2 \\ 4 &= 2+2 \\ 4 &= 1+3 \\ 4 &= 4 \end{aligned}$$

We write $p(n)$ for the number of partitions of n. We define $p(0) = 1$, and see that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, etc. The sequence $p(n)$ determines the *generating function*

$$F(z) = \sum_{n=0}^{\infty} p(n) z^n \ . \tag{2}$$

If the sum converges, the function $F(z)$ contains information about the numbers $p(n)$.

(a) Show that the radius of convergence of the power series is at least $r = \frac{1}{2}$ by showing that $p(n) \leq 2^{n-1}$. *Hint.* 2^{n-1} is the number of subsets of $n-1$ things, which can be spaces between integers k and $k+1$. It is possible find a unique subset of these spaces corresponding to any partition. (Arguments like this are called *combinatorial* – deriving equalities or inequalities between combinatorial numbers by constructing functions between collections of objects.)

(b) Prove the infinite product formula of Euler, which is true if $|z| < 1$:

$$F(z) = \frac{1}{1-z} \frac{1}{1-z^2} \frac{1}{1-z^3} \cdots = \prod_{n=1}^{\infty} \frac{1}{1-z^n} \ . \tag{3}$$

This is true "formally" because

$$\begin{aligned} \frac{1}{1-z} \frac{1}{1-z^2} \cdots &= \left(1 + z + z^2 + \cdots\right) \left(1 + z^2 + z^4 + \cdots\right) \cdots \\ &= \sum_{n=0}^{\infty} a_n z^n \ . \end{aligned}$$

The coefficient a_n is the number of ways $z^n = z^{k_1} z^{k_2} \cdots$ can arise using z^{k_1} from the first series $(1 + z + z^2 + \cdots)$, z^{k_2} from the second series $(1 + z^2 + z^4 + \cdots)$, etc. Think of k_1 as the number of ones in the partition of n, k_2 as the number of twos, etc. This exercise asks you to use these ideas to construct a real proof.
Hints. First prove that the product converges absolutely if $|z| < 1$. Then consider finite products

$$F_M(z) = \frac{1}{1-z} \cdots \frac{1}{1-z^M} = \sum_{n=1}^{\infty} q_M(n) z^n \ . \tag{4}$$

The numbers $q_M(n)$ represent the number of partitions of n into "pieces" not larger than M. For example, $q_2(4) = 3$, corresponding to the first three partitions listed above. Clearly, $q_M(n) \leq p(n)$, $q_M()$ is an increasing function of n (why?), and $q_M(n) = p(n)$ for large M (why?). For $|z| < \frac{1}{2}$, the original power series (2) converges and the "sub-series" converge to it. Next, the Euler product (3) converges for $|z| < 1$, so $F(z)$ is an analytic function with a Taylor series that converges for $|z| < 1$ (why?).

(c) Show that for any $\epsilon > 0$ there is a C_ϵ so that

$$p(n) \leq C_\epsilon e^{\epsilon n} \ , \quad \text{for all } n \ . \tag{5}$$

(d) *Extra credit, don't waste too much time with this.* Find a proof of (5), or any inequality of the form $p(n) \leq CA^n$ with $A < 2$, that does not use complex analysis.