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Assignment 2, due September 17

Corrections: (September 12) “put ← call” in question 1. “return” ← “cash flow” in question 2i.

1. Show that the price of a vanilla European call or put is a convex function of the strike price. *Discussion:* A function $f(K)$ is a convex function of K (in the interval $[K_0, K_3]$) if

$$f(\alpha K_1 + (1 - \alpha)K_2) \leq \alpha f(K_1) + (1 - \alpha)f(K_2), \quad (1)$$

whenever $0 \leq \alpha \leq 1$ (and $K_0 \leq K_1 \leq K_2 \leq K_3$). If $f(K)$ is twice differentiable as a function of K , this is the same as $f''(K) \geq 0$. We can check (1), say, for $\alpha = 1/2$, by comparing the payout diagram for the portfolio:

$$\frac{1}{2}(\text{call at } K_1) + \frac{1}{2}(\text{call at } K_2)$$

to the payout of the single call struck at $\frac{1}{2}(K_1 + K_2)$ (draw the diagram). The argument for other values of α and for puts is similar.

Some options dealers post a curve of options prices at which they are willing to buy and sell as functions of strike and maturity. If the curve is not convex for any reason (e.g. non-convex interpolation) this represents an arbitrage opportunity for the client.

2. In a one period model, suppose a risky asset has a present spot price $S_0 = 1$. The unknown spot price after one period is S_1 . There is a $p_u = 90\%$ chance that $S_1 = s_u = 3$ and a $p_d = 10\%$ chance $S_1 = s_d = 1$. A risk free dollar invested today yields two dollars after one period (technically, $B(0, 1) = \frac{1}{2}$). Let C be the payout of a vanilla European call option, with strike price $K = 1.5$ that may be exercised after one period. It is common to refer to probabilities in the real world as the “P measure” and probabilities in the risk neutral world as the “Q measure”.
 - (a) What is the expected discounted payout of the call option: $E_P [B(0, 1) C]$?
 - (b) What are the risk neutral probabilities q_u and q_d , of $S_1 = 3$ and $S_1 = 1$ respectively? Is $p_u > q_u$?
 - (c) What is the expected discounted value of C with respect to the risk neutral probabilities, $E_Q [B(0, 1) C]$? This is the arbitrage price of C .
 - (d) Is the arbitrage price larger or smaller than the expected discounted payout from part (b)? Why?

- (e) Let Π be the portfolio at the present time that exactly hedges C . How much risk free asset and how much risky asset are in Π ?
- (f) If you buy the option at its risk neutral price, what are the mean and standard deviation of your expected return? How does this compare to the mean and variance of the return on the stock? Make sure to make the return a percentage – money gained (or lost) divided by money invested.
- (g) If we fix the value of s_d , what value of s_u results in the P probabilities being equal to the Q probabilities? Call this s_u^* .
- (h) Suppose it were possible to buy or sell the call option today at the price equal to its real world expected value from part (a). Describe the arbitrage portfolio that profits from this possibility, being short or long one unit of C . Call this portfolio Π_A .
- (i) Suppose that S_1 is a Gaussian random variable that matches the real world mean and variance: $E_G[S_1] = E_P[S_1]$ and $E_G[S_1^2] = E_P[S_1^2]$. (Here, $E_G[\cdot \cdot \cdot]$ refers to expected value if S_1 is Gaussian.) What are the expected discounted cash flow and variance: $E_G[B(0,1)\Pi_A]$ and $\text{var}_G[B(0,1)\Pi_A]$?
3. Consider a one period market in which the forward price is described by a trinomial tree/ The forward price today is $\mathcal{F}_0 = 10$. The possible forward prices at time T are $\mathcal{F}_d = 8$, $\mathcal{F}_m = 11$, and $\mathcal{F}_u = 12$. Take the risk free rate to be so that $e^{rT} = 2$. Consider an option whose payout is $V_d = 0$, $V_m = 10$, and $V_u = 15$. The price of the option today is V_0 .
- (a) Show that it is not possible to replicate V_T (one of the three numbers V_d, V_m, V_u) exactly with the forward and cash. Conclude that there is more than one value of V_0 that does not permit arbitrage with cash and the forward.
- (b) Is there a value of V_m so that V_0 is uniquely determined by the no arbitrage condition? If so, find it.
- (c) Clearly $0 \leq V_0 \leq 7.5$. Use the arbitrage construction to get tighter bounds.