Assignment 3, due September 24

Corrections: September 20, Problem 1: $B(0,T) = .5$ corrected to $B(0,T) = .95$, the hedge components corrected to $S^{(1)}$ and $S^{(2)}$.

1. There are two stocks, $S^{(1)}$ and $S^{(2)}$. Both are at 100 today: $S_0^{(1)} = S_0^{(2)} = 100$. A "guessed right" option pays $G = \max\left(S_T^{(1)}, S_T^{(2)}\right)$ at time T. Stock $S^{(1)}$ may go to 110 or 90 at time T. Stock $S^{(2)}$ may go to 130 or 80 at time T. The risk free rate is so that $B(0,T) = .95$. All combinations of possible values of $S_T^{(1)}$ and $S_T^{(2)}$ are possible. Is it possible to make a riskless hedge of G using $S^{(1)}$, $S^{(2)}$, and the bond? If so, what is the arbitrage price of G at time $T = 0$. If not, why not?

2. We want to price a put option that expires in one year and has strike price 120 (at the money forward). Use a three period binomial tree model with a fixed geometric ratio of a 20% upside and 10% downside for each of the three periods. That means if S_t is the stock price at time t, then $S_{t+\delta t}$ may be either uS_t or dS_t with $u = 1.2$ and $d = .9$. The initial stock price is $s_0 = 100$. The risk free return is $r = 20\%$/year (so that one dollar at time $t = 0$ becomes $e^{.2 \cdot 1} \approx 1.22$ after a year at the risk free rate). The time step for each period is $\delta t = \frac{1}{3}$year.

   The questions below will require a certain amount of arithmetic. Please do the arithmetic by hand or with a calculator. Do not write a computer program or even program Excel to do the calculations.

   (a) Draw the full three period tree. At each node, indicate the stock price, the option price, the amount of cash (bond), and the amount of stock appropriate for that node in order to make the risk free hedge.

   (b) Let uud represent the path $s_0 \to us_0 \to u^2 s_0 \to u^2 d s_0$. How much stock must be bought or sold at the end of each period for this path?

   (c) Repeat part (b) for the path udu. Note that the paths are different and there will be different cash and stock positions along the path, but they end at the same price at the end of the final period.

   (d) Calculate the risk neutral probabilities for each of the outcomes $S_1 = u^3 s_0$, $S_1 = u^2 d s_0$, $S_1 = u d^2 s_0$, and $S_1 = d^3 s_0$. Here, S_1 is the stock price when $T = 1$year. The probability of reaching a given value of S_T is determined by the probability of an individual path and the number of paths.

(e) Evaluate $e^{-rT}E_Q\left[f(S_T)\right]$, where $T = 1$year and $E_Q[\cdot]$ is the risk neutral probability. Do this using the four probabilities calculated in part (d). Check that this answer agrees with the option price computed in part (a).

3. Consider a geometric binomial tree model with time steps δt and possible moves $s \to us$ or $s \to ds$ on one period, and risk free rate r, as in problem 2.

   (a) Show that $E_Q\left[S_{\delta t}\right] = s_0 e^{r\delta t}$.

   (b) Show that $E_Q\left[S_T\right] = s_0 e^{rT}$ for any number of time steps: $T = k\delta t$. Hint: binomial coefficients are not the easiest way to do this.

   (c) Define $M_t = e^{-rt}S_t$. Suppose $T = k\delta t$ and $T' = n\delta t$ are two times in the tree, with $T' > T > 0$. Show that

$$E_Q\left[\, M_{T'} \mid M_T = m_* \,\right] \;=\; m_* \,. \tag{1}$$

   The notation $E[X \mid A]$ refers to the expected value of the random variable X conditioned on the event A happening. In this case, we seek the expected value of $M_{T'}$ conditioned on $M_T = m_*$. A random process, M_T, that satisfies (1) is called a *martingale*. The formula (1) expresses the fact that in the risk neutral measure, the stock price becomes a martingale when discounted at the risk free rate.