Derivative Securities, Courant Institute, Fall 2008
http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec08/index.html
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Assignment 6, due October 22

Corrections: (Oct. 19: Problem 1 clarified – the U_k independent of each other, Problem 2 – assume $f(t)$ has finite variance for each t. Oct. 21: the hypotheses of Problem 2d clarified, the SDE in problem 4 corrected to put $S(t)$ in the noise term.)

This assignment is really Stochastic Calculus. Applications to finance are coming. It is very important to understand the two basics: (i) why dW^2 is important and in what sense $(dW)^2 = dt$, and (ii) the relation between partial differential equations (PDE) and stochastic differential equations (SDE).

(For Problem 1 and Problem 2) Suppose $f(t)$ is a random function of t – a Brownian motion or an Ornstein Uhlenbeck path or something like that. We say that $f(t)$ is *adapted* (or *non-anticipating*, which means almost the same thing) if the *future path*, $W(t+s) - W(t)$, for $s > 0$, is independent of all the values $f(s)$ for $s \leq t$. This is the same as saying that as much as $f(t)$ is influenced by values of W, it is influenced by values of $W(s)$ for $s \leq t$. For example,

$$f(t) \; = \; \int_0^t W(s)\, dW(s) \; = \; \frac{1}{2}\left(W(t)^2 - t\right) \; ,$$

where $f(t)$ depends only on $W(t)$, and $f(t) = X(t)$, the solution of $dX = -aX dt + \sigma dW$:

$$f(t) \; = \; e^{-at}X(0) \; + \; \sigma \int_0^t e^{-a(t-s)}\, dW(s) \; .$$

In the second example, $f(t)$ is not completely determined by W because it depends also on $X(0)$, which could be a different random variable. Also, $f(t)$ depends on the whole path $W(s)$ for $s \in [0,t]$, not just the value $W(t)$. In both cases, $f(t)$ has nothing to do with $W(s)$ for $s > t$.

1. Suppose F_k for $k \leq n$ are a family of random variables, and U_k are other random variables with U_k independent of $F_1, \ldots, F_k$ and independent of U_j for $j \neq k$. Suppose that $E[U_k] = 0$ and $E[U_k^2] = \sigma_k^2$. Let $A = \sum_{k=1}^n F_k U_k$.
   Show that
   $$E\left[A^2\right] \; = \; \sum_{k=1}^n \sigma_k^2 \, E\left[F_k^2\right] \; . \qquad (1)$$
   Relate this to the first formula on page 1 of Kohn's notes, Section 7, the *Ito isometry formula*.

2. (This is much like Kohn's discussion of discrete time rebalancing starting on page 3. It's n explanation of $dW^t = dt$.) Suppose $f(t)$ is a random adapted function. Consider the approximation

$$\int_0^T f(t)\,(dW(t))^2 \;\approx\; \sum_{t_k \leq T} f(t_k)\big(\,W(t_{k+1}) - W(t_k)\,\big)^2 \;,$$

where δt is a small time step and $t_k = k\delta t$. Mathematically, this means that the right side has a limit as $\delta t \to 0$ (with T fixed), and that limit is the definition of the left side. (Assume that $E[f(t)^2] < \infty$ for each t.)

(a) Define

$$U_k \;=\; \big(\,W(t_{k+1}) - W(t_k)\,\big)^2 - \delta t \;.$$

Calculate $E[U_k]$ and $E[U_k^2]$.

(b) Define

$$A \;=\; \sum_{t_k \leq T} f(t_k) U_k \;,$$

so that

$$\sum_{t_k \leq T} f(t_k)\big(\,W(t_{k+1}) - W(t_k)\,\big)^2 \;=\; \sum_{t_k \leq T} f(t_k)\delta t \;+\; A \;. \qquad (2)$$

Explain why the hypotheses of Problem 1 apply to this A.

(c) Use formula (1) to get a formula for $E[A^2]$.

(d) Suppose that $E\left[f(t)\right] \leq M$ and $E\left[f(t)^2\right] \leq M$ for all $t \leq T$ and that $f(t)$ is a continuous function of t. Show that A is of the order of $\sqrt{\delta t}$, at least in the mean square sense, as $\delta t \to 0$.

(e) Use this to argue that in the limit $\delta t \to 0$ we get

$$\int_0^T f(t)\,(dW(t))^2 \;=\; \int_0^T f(t) dt \;. \qquad (3)$$

To summarize: The formula $(dW)^2 = dt$ is true only in the average sense that $E[(dW)^2] = dt$. However, the fluctuations are small enough that they cancel out in the sum (3).

(f) (not action item) This exercise illustrates a feature of stochastic calculus. We prove things, such as $A \to 0$ as $\delta t \to 0$ by finding things we can calculate, here $E[A^2]$.

3. The Ornstein Uhlenbeck mean reverting process is

$$dX(t) \;=\; -aX(t)dt \,+\, \sigma dW(t) \;. \qquad (4)$$

(a) Derive a formula for $X(T)$ in terms of $X(t)$ and the values of $W(s)$ for $t \leq s \leq T$. This formula will be similar to the one in the notes.

(b) Use the formula from part (a) to evaluate

$$f(x,t) \;=\; E_{x,t}\left[\, X(T)^2 \,\right] \;=\; E\left[\, X(T)^2 \mid X(t) = x \,\right] \; .$$

(c) The general backward equation (not written explicitly in Kohn's notes) is

$$\partial_t f \;+\; \frac{1}{2} b^2(x,t)\partial_x^2 f \;+\; a(x,t)\partial_x f \;=\; 0 \; ,$$

where $f(x,t) = E_{x,t}[V(X(T))]$, $f(x,T) = V(x)$ (the *final conditions*, and $dX = a(X(t),t)dt + b(X(t),t)dW(t)$. Check that the result of part (b) satisfies the backward equation and the backward equation for the mean reverting process (4) and the correct final conditions.

(d) Solve the backward equation for (4) that with final conditions $f(x,T) = x^4$. Hint: at each time, t, this is a polynomial of degree 4. You need only find the coefficients as functions of t. Check that this is true.

(e) Evaluate $E[X(T)^4 \mid X(0) = 0]$ using the fact that $X(T)$ is a Gaussian with mean zero and known variance (in the notes) and the fact that if Y is normal with mean zero, then $E[Y^4] = 3\sigma_Y^4$, where $\sigma_Y^2 = \mathrm{var}(Y)$.

4. A *volatility surface* is a function $\sigma(s,t)$. A volatility surface model of stock price movement is

$$dS(t) \;=\; \mu S(t)dt \;+\; \sigma(S(t),t)\, S(t)\, dW(t) \; .$$

(a) Use the $\Delta-$hedging argument of Black and Scholes outlined in Kohn's notes, Section 6 and 7 (not the binomial tree argument) to derive a PDE that prices European options for a volatility surface model.

(b) Use the backward equation relationship to derive the risk neutral process (an SDE for $S(t)$) so that this price is $E_{s,t,RN}[V(S(T))]$.

(c) The realized return variance for time T and time step δt is

$$R \;=\; \sum_{t_k \leq T} \left(\frac{S(t_{k+1}) - S(t_k)}{S(t_k)} \right)^2 \; .$$

Find a formula involving an integral for the limit of the realized variance as $\delta t \to 0$ in the stochastic volatility model.