Derivative Securities, Courant Institute, Fall 2006
Sample Final Exam
Questions added December 15, 4pm.
Question 2 (Feynman Kac) revised and sign corrected December 18, 12:30

Important:

- The final exam is on Wednesday, December 20, from 7:10 to 9 pm in the usual classroom.
- You only may bring one piece of paper with whatever information you want to put on it. You must be able to read this without magnifying lenses or electronics. No other materials or books are allowed.
- You may not use and will not need a calculator.
- The questions below are examples of the kind of questions that will be on the actual final. The questions on the final will be different, though some may be similar.
- I will have special office hours from 5 to 7 on Monday and Tuesday, December 18 and 19.
- Feel free to post and argue over answers on the class bboard .

Part 1, multiple choice

1. I have a formula for $Y(t)$, the effective interest rate for money borrowed now and repaid at time t. This formula is called the (see Hull, Chapter 28)
   (a) Volatility of interest rates (b) Term structure of interest rates
   (c) Option price of interest rates (d) Hedge ratio of interest rates

2. In the martingale measure with $Y(t)$ to be a numeraire for a one factor market with tradable instruments $X_1(t)$, ..., $X_n(t)$ with no arbitrage opportunities, what must be constant in time: (Hull, Chapter 25)
   (a) The volatility of $Y(t)$ (b) The expected return of $Y(t)$
   (c) The expected value of $Y(t)/X_1(t)$ (d) The expected value of $X_1(t)/Y(t)$

3. The term *volatility skew* refers to the fact that

   (a) The implied volatility of an option is an increasing or decreasing function of the strike price

   (b) Different stocks have different volatilities

   (c) The volatility is an increasing function of time

   (d) The volatility is a convex function of the strike price.

4. The *smooth pasting* condition for American style options is the fact that

(a) The Gamma of the option is a smooth function of the strike price

(b) The early exercise boundary is a smooth function of time

(c) The Delta of the option is ± 1 (call or put) at the early exercise point

(d) The early exercise boundary converges to the strike price as time approaches the expiration time.

5. Which data would we be most likely to use to construct a yield curve?
(a) NASDAQ (b) the London EDX
(c) FEMA (d) LIBOR

6. We use the risk neutral measure for pricing stock options because

   (a) Large institutions have so much capital that they are insensitive to risk

   (b) It is easier to estimate the risk neutral measure than the historical measure from historical price data

   (c) Risk neutral prices are generally lower, so we can enter options contracts at less cost

   (d) Arbitrage pricing theory suggests that future option prices are determined by the risk neutral measure.

7. A loan is being negotiated under which the counterparty will pay the one month (answer to question 5) rate each month for five years and then repay the principle. One of the parties asks that if that rate is more than 7% (annualized), then he should pay 7% instead. This extra aspect of the loan contract is called
(a) a cap (b) a limit order
(c) a swap (d) a forward contract

8. I believe that the price of a stock is going to change only a little relative to the implied volatility implicit in the near the money puts and calls. The price is S. To profit from that belief (if true) I should (draw the payout diagrams and name them)

   (a) buy one share of stock and short Δ shares of the at the money option

   (b) buy one put with strike $K_1 < S$ and one call with strike $K_2 > S$

   (c) buy a call struck at $K_1 = S - \widetilde{S}$, (with $\tilde{S}$ small but positive), buy a put struck at $K_2 = S + \widetilde{S}$, and sell a call and a put both struck at S.

   (d) buy a call struck at $K_1 < S$ and sell a put struck at $K_2 > S$.

9. We use the binomial tree method for option pricing vanilla European options instead of the Black Scholes formula because

   (a) the binomial tree model for stock price changes is more realistic than geometric Brownian motion

   (b) people who don't know calculus can understand it

(c) it takes the computer less time to do a binomial tree than to evaluate the Black Scholes formula

(d) the arbitrage argument for geometric Brownian motion (the Black Scholes argument) is less rigorous than the CRR hedging argument using binomial trees.

10. The Gamma of a vanilla American style option is greatest (Fire up one of your old homework codes to figure this out.)

    (a) near the strike price, close to expiration

    (b) deep out of the money, close to expiration

    (c) near the strike price, far from expiration

    (d) near the early exercise boundary

Part 2

1. Someone in country A will know in six months whether she needs to aquire one unit of currency B one year after that, 18 months from now. Suppose $S(t)$ is the price in currency A of one unit of currency B at time t. She wants to pay K units of A to get one unit of B, if she aquires the forward contract. She wants an option to aquire the forward contract in six months. Wue a geometric Brownian motion model of $S(t)$ with mean expected growth parameter μ and volatility σ. Denote by r_A and r_B the interest rates in the respective countries for funds in the respective currencies. Give the appropriate Black-Scholes formula for the present price of this option in terms of r_A, r_B, σ, μ (if needed), $t = .5$ from today years to enter into the forward contract, and $T = 1.5$ years from today to pay K units of currency A for a unit of currency B. *Hint:* This stuff is discussed nicely in Chapter 14 of Hull (sixth edition) and in the Kohn and Allen notes.

2. We want a simple mean reverting interest rate model that never has negative interest rates. For that reason we model $X(t) = \sqrt{r(t)}$, so that $r(t) = X^2(t)$ Suppose $dX = a\left(\overline{X} - X\right)dt + \sigma dW$ and we want to evaluate

$$f(x,t) = E_{x,t}\left[\exp\left(-\int_t^T X^2(s)ds\right)\right] .$$

   Use the Feynman– Kac formula to write a PDE we could solve to find f. Give the final conditions. What market instrument would this be the value of (when $t = 0$)? Use x, t as independent variables for the PDE, not r.

3. Suppose we have data points P_k which are the prices of bonds that expire in k years, pay a coupon of C each year (one payment at the end of each year) before year k and one dollar at year k. We have these numbers

for $k = 1, 2, \ldots, n$. Write a piece of a VBA program that computes the numbers $B(0,k)$, that are the implicit prices of zero coupon bonds with the same maturities (see Chapter 4 of Hull). Assume that we already have an array `dim P(n) as Double` that holds the numbers P_k and we want to put the answers into another array `dim B(n) as Double` so that `B(k)` holds the number $B(0,k)$. What is this process called?

4. In the world where the money market is the numeraire, a stock price $S(t)$ is a geometric Brownian motion and the short rate, $r(t)$, satisfies the CIR model, write a formula for the price of a European call option (expiration T and strike K) of the form

$$f(s,r,0) = E\left[\text{ something involving the path } r(t) \text{ and } S(T)\ \right] \quad . \tag{1}$$

Hull, Section 25.4, says how to do this. Let $r[t,T]$ be the path $r(s)$ for s in the range $t \leq s \leq T$. Write a generalization of (1) for the price $f(s,r,t)$ in terms of $S(T)$ and $r[t,T]$. The r argument of f is $r(t)$. Use the Feynman Kac formula to write a two dimensional PDE satisfied by f. State the final conditions satisfied by $f(s,r,T)$. The parameters are σ_1, volatility of the stock price process, σ_2, the volatility r, $\overline{r}$, the mean reversion point for the short rate, a, the mean reversion rate for the short rate, and ρ, the correlation between the Brownian motions driving S and r.

5. Let $f(s,t)$ be the price of a down and out American style put option. The strike price is K and the knockout price is $L < K$. Explain a forward Euler/trinomial tree method for evaluating f. What is $f(L+0,t)$, the limiting value of option at the option price as S approaches L from above. Warning: the answer is intuitive but I don't know a way to derive it rigorously. Maybe someone will find a way, but it is not necessary. Just explain your reasoning.