Assignment 1, due September 23

Corrections: (Sept 10: Change of due date. Sept 14: S&P 100 corrected to S&P 500 in question 1.)

1. A medium scale investor wants to invest in the US equity market. She wants the return to be very nearly that of the S&P 500. Suggest, in a few clear sentences, several ways she might do this. Discuss the advantages and disadvantages of an ETF, a futures position, or just buying the assets in the index directly. For the ETF and futures approaches, name the specific asset and the exchange it trades on.

2. A company needs to borrow money at time t to be repaid at time T but wants to fix the price today. The notation for this is that company agrees today (time $t = 0$) receive $F(0,t,T)$ at time t and pay one unit of currency at time T.

   (a) Explain an arbitrage argument that fixes $F(0,t,T)$ in terms of $B(0,t)$ and $B(0,T)$. Hint: in the absence of market frictions and counterparty risk, the company can go long a value X of the $B(0,t)$ and short the same value of the $B(0,T)$ asset.

   (b) If interest rates fluctuate, can the company guarantee that the amount it pays for the loan, $F(0,t,T)$, will be equal to the spot price of the same loan when it is taken out at time t?

3. Give an arbitrage argument to show that the price of a vanilla European call or put is a convex function of the strike price, if the exercise time is held fixed. Assume that there are no market market frictions or trading constraints. *Discussion:* A function $f(K)$ is a convex function of K (in the interval $[K_0, K_3]$) if

$$f(\alpha K_1 + (1-\alpha)K_2) \leq \alpha f(K_1) + (1-\alpha) f(K_2) \;, \qquad (1)$$

   whenever $0 \leq \alpha \leq 1$, and $K_0 \leq K_1 \leq K_2 \leq K_3$. If $f(K)$ is twice differentiable as a function of K, this is the same as $f''(K) \geq 0$. We can check (1), say, for $\alpha = 1/2$, by comparing the payout diagram for the portfolio:

$$\frac{1}{2}(\text{ call at } K_1 \,) + \frac{1}{2}(\text{ call at } K_2 \,)$$

to the payout of the single call struck at $\frac{1}{2}(K_1 + K_2)$ (draw the diagram). The argument for other values of α and for puts is similar.

Some options dealers post a curve of options prices at which they are willing to buy and sell as functions of strike and maturity. If the curve is not convex for any reason (e.g. non-convex interpolation) this represents an arbitrage opportunity for the client.

4. Find a formula for the variance of a lognormal ransom stock price as a function of the parameters μ, σ, S_0, and T.

5. This part asks you to do some numerical computations using Microsoft Excel. The file `ExcelHelp.pdf` on the class web site will help you do what this assignment asks for. This assignment addresses a point made very clearly by Nassim Taleb in his book *Fooled by Randomness.* That is that fund managers can skew their return distribution so that most years they outperform some benchmark, but some years they "blow up". If the probability of blow up is, say, 10%, the manager can have a ten year run of high returns. Getting fired when the fund blows up is a small price to pay for all those bonuses received in the mean time. How to "incentivize" managers not to do this is a lively topic among regulators, businesses, and investors.

   (a) The random number generator in Excel has proven (in my tests) to be inadequate for the steps below. Do not use it. Instead, download the file `randn.xls` from the class web site. It has 5000 standard normal random variables created by a decent generator.

   (b) Use the standard normals to generate sets of a thousand independent lognormal stock prices, with vol parameter σ, expected rate of return μ, and time horizon T. The formula is

   $$S \;=\; S_0 \exp(\sigma Z \sqrt{T} + (\mu - \sigma^2/2)T) \;, \qquad (2)$$

   where Z is a standard normal. Verify that $E[S] = S_0 e^{\mu T}$ to within the accuracy of Monte Carlo. If you have n independent standard normal random variables $Z_1, \ldots, Z_n$, you can make n independent lognormals using (2), use Z_k to make S_k. Then you can estiamte $E[S]$ as $\frac{1}{n}\sum_{k=1}^{n} S_k$. Use $n = 1000$ here, $\sigma = .2$, $\mu = .1$, $S_0 = 100$, and $T = 2$. Create a histogram You will learn much more about Monte Carlo computation like this going forward in the math finance program.

   (c) Consider a vanilla European call option on S with strike price K that expires also at time T. Let C be the price of this option. Estimate C using the Black Scholes lognormal model/theory and Monte Carlo. Use the $n = 1000$ stock prices S_T generated in part (b). Use $r = .03$ and $K = 130$.

(d) Consider a portfolio strategy that is to buy one share of stock and to sell short M call options at the initial time. The strategy holds the stock, invests the money from selling the option at the risk free rate. At time T, the portfolio value is $S_T - MV(S_T) + Ce^{rT}$. With the parameters above, it is unlikely that the option will finish *in the money* ($V(S_T) > 0$). The most likely outcome is that the final value of the portfolio with the option is equal to $S + Ce^{rT}$. This outperforms the simple stock portfolio by Ce^{rT}. Create one graph that has histograms of the two portfolios, one with and one without the option.

(e) Which portfolio has the greatest expected value at time T?

(f) Which portfolio is most likely to outperform the other?

(g) Explain in a few sentences how this illustrates Taleb's point.