

Assignment 10, due December 2, not November 24

Corrections: (several, Nov 25)

1. This exercise and the next one solves one of the simplest interest rate models in a sequence of steps. The model is that in the “world” where the money market is the numeraire, the short rate satisfies the SDE

$$dr_t = a(\bar{r} - r_t)dt + \sigma dW_t. \quad (1)$$

This is an *equilibrium* model because the probability distribution of r_t has a limit, an *equilibrium*, as $t \rightarrow \infty$. It seems clear that in this limit $E_{r_0}[r_t] \rightarrow \bar{r}$. We will see that this model makes many specific predictions, including possible shapes of the yield curve and an *affine* pricing model for bonds and certain interest rate derivatives. All this follows from the mathematics of Gaussian random variables, and the fact that (1) is a Gaussian model.

It is a sad fact in finance that many interest rate model formulas and calculations are a little messy. Part of this exercise and the next one is to find a way to express the formulas and computations that makes them manageable. This may mean defining intermediate quantities, such as r_t^* below. Feel free to define your own notation in a way that makes these computations and formulas as clear and straightforward as possible for you.

- (a) Find a solution to (1) of the form

$$\bar{r} - r_t = f(t)(\bar{r} - r_0) + \int_0^t g(t-s)dW_s, \quad (2)$$

where $f(t)$ and $g(t)$ are simple non random exponentials. Rewrite this as

$$r_t = r_t^* + R_t, \quad (3)$$

where r_t^* is not random and R_t always has mean zero. Give formulas for r_t^* and R_t as functions or explicit integrals.

- (b) Using a finite difference approximation if necessary, argue from part (a) that r_t is a Gaussian random variable with known mean r_t^* and

$$\text{var}[r_t] = E \left[\left(\int_0^t g(t-s)dW_s \right)^2 \right] = \int_0^t g(t-s)^2 ds. \quad (4)$$

Part of the problem is to show that the r_t has Gaussian distribution. The other part is to verify the formulas for its mean and variance. The second equality of (4) is a special case of the *Ito isometry formula* of stochastic calculus.

- (c) Calculate the integral in (4) to get a closed form formula for $\text{var}[r_t]$.
- (d) Suppose M_t is the *money market* variable that satisfies $dM_t = r_t M_t dt$, $M_0 = 1$. Find a formula for M_T of the form

$$M_T = e^{Q_T} ,$$

where

$$Q_T = \int_0^T r_t dt = q_T^* + \int_0^T k(T, s) dW_s . \quad (5)$$

Here $q_T^* = E[Q_T]$. Find an explicit formula for q_T^* .

- (e) Repeat steps (b) and (c) to show that Q_T is Gaussian and to get an explicit formula for variance of Q_T .
- (f) Argue that $B(t, T)$ (the price paid at time t for one unit of currency at time $T \geq t$) is tradable at time t , and therefore that $B(t, T)/M_t$ is a martingale (recall what world this problem is set in). Use this to deduce that $B(0, T) = E[e^{-Q(T)}]$, where Q_T is the quantity in (5). This argument is in Chapter 27 of Hull.
- (g) Use your known formulas for the mean and variance of Q_T to find formula of the form

$$B(0, T) = e^{A_T + r_0 B_T} . \quad (6)$$

Find explicit formulas for A_T and B_T . (Be careful of the almost conflict of notation: the B_T in the exponent on the right is not the $B(0, T)$ on the left.) This is called an *affine model* because the exponential on the right side of (6) is an affine function of r_0 . The coefficients in the affine model, A_T and B_T , depend on all the parameters in the problem (including T), but not on r_0 . What kind of random variable does this make $B(0, T)$?

- (h) Use (6) and your explicit formulas for A_T and B_T to find an explicit formula for the yield curve ($e^{-TY_T} = B(0, T)$)

$$Y_T = Y(T, r_0, \bar{r}, a, \sigma) . \quad (7)$$

- (i) Use trial and error, or the Excel solve capability if you are adventurous, to identify what parameter values \bar{r} , a , and σ give the best fit (or a good fit) of the actual yield curves based on Treasuries from assignment 9. Use $r_0 = 0$ (or $r_0 = .03$ if you insist). Explain how you use rough properties of your formula for Y_T to get rough estimates of the parameters by looking at the yield curve numbers.

- (j) Make a plot showing your best yield curve fit and the actual yield curve data. Use any of the Treasury yield curves from homework 9 that you wish, as they are very similar for this purpose.
- (k) Make plots of some yield curves that arise from other values of r_0 , \bar{r} , a , and σ . Are there some shapes that are not allowed? Is it possible to produce an inverted yield curve?
2. The methods above allow us to derive Black Scholes type formulas for the values of bond options in the model (1). We will treat only the simplest case, the value of a European style call option on the short rate. This option pays $(r_T - K)_+$ at time T . Let $f(r, t)$ be the value/price of the option at time t if $r_t = r$. If the formulas start to get too long and messy (you decide), try to simplify your exposition of them by defining intermediate quantities. We seek a recipe that could be easily implemented in a few Excel formulas if a single formula is too complicated.
- (a) Derive a formula of the kind $f(r, t) = E_{r,t} [e^{-Q_{t,T}} (r_T - K)_+]$. Find a formula for $Q_{t,T}$ in terms of r_t and the path W_s for $t \leq s \leq T$. One way to do this is to repeat much of Question 1, but starting with $r_t = r$ instead of $r_0 = r_0$.
- (b) From now on, assume $t = 0$. Show that r_T and Q_T are jointly normal with covariance (using notations from question 1)

$$\text{cov}(r_T, Q_T) = \int_0^T g(T-s)k(T,s)ds .$$

Evaluate this integral explicitly.

- (c) We need the following general fact about correlated normal random variables. Suppose that the joint distribution of X and Y is a two dimensional normal with $\sigma_X^2 = \text{var}(X)$, $\sigma_Y^2 = \text{var}(Y)$, and $\sigma_{XY} = \text{cov}(X, Y)$. Find the formula for a constant C so that $Y = CX + Z$ where Z is a normal independent of X (not just uncorrelated). Find a formula for $\sigma_Z^2 = \text{var}(Z)$.
- (d) Apply the decomposition method of part (c) to $X_T = r_T$ and $Y_T = Q_T$. Show that this gives

$$f_0 = E [e^{-Z_T}] E [e^{-C_T r_T} (r_T - K)_+] .$$

Evaluate $E [e^{-Z_T}]$.

- (e) Give a formula for $E [e^{-C_T r_T} (r_T - K)_+]$ in terms of a cumulative normal function and another term. Review our derivation of the Black Scholes formula to see how to do this. Use this to give a formula for f_0 in the spirit of the Black Scholes formula.
- (f) *Without doing more complicated algebra*, explain how you would find a similar pricing formula for an option on a zero coupon bond. That

would be the price today for the option at time T to receive K at time T and repay one unit at time $T_1 > T$. What properties of the pricing formula (from question 1) for $B(T, T_1, r_T)$ are necessary to make this work?

3. (Deleted)

4. Consider three bonds:

coupon	maturity	price
none	six months	98.5
3%, semi-annual	one year	99.7
4%, semi-annual	two years	101.9

Use the bootstrapping method, possibly implemented in Excel, to determine the six month, one year, and two year yields implicit in these prices, which are given on a par value of 100.

5. (Extra credit. Read section 27.1 carefully first.) Show that the world in which the money market is the numeraire is the risk neutral world – the one in which the market price of risk is zero. What is the market price of risk in the world in which $B(t, T)$ is the numeraire? In particular, is it a constant? When you reweight to make $B(t, T)$ the numeraire, how does the model (1) change?