Derivative Securities, Courant Institute, Fall 2009

http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 11, due December 9, with a penalty free grace period of 1 week.

Corrections: (none yet)

1. This exercise will use Black's method to get a price for the following cap. A company has a floating rate $\$10^7$ loan with the following features. At the end of each six month period, it must pay "LIBOR + 2" on the principal, which means the six month LIBOR rate at the beginning of that period plus 2%/year. The company wants a cap on its interest rate payments at 7%. That is, it wants its semi-annual payments never to be higher than a 7% annual rate (3.5% each half year). Carry out the following steps, using Excel for actual calculations.

   (a) As described in Hull, this cap may be understood as a sum of twelve *caplets*. Describe each of these caplets as a European style put or call option on a zero coupon bond. What is the bond for caplet k and what is the strike?

   (b) At the present time, $B(t,T)$ is unknown. Show that any short rate model in the risk neutral world gives $B(0,t)E[B(t,T)] = B(0,T)$. Note that the left side corresponds to the strategy of getting one unit of currency at time T by buying a bond today that matures at time $t < T$ then buying a bond at an unknown price at time t that matures at time T. Hint: in the risk free world, the money market is the numeraire and we have a formula for bond prices in terms of $E\left[\exp(\int r_s ds)\right]$.

   (c) Assume that $B(t,t_1)$ (with $t_1 > t$) is lognormal with expected value given by part (b) and $\text{var}(\ln(B(t,t_1)) = \sigma(t_1 - t)t$. Choose σ so that the standard deviation of a one year bond starting in one year (maturing two years from now) is fifty basis points. Take $B(0,t)$ so that the yield curve is given by the formula $Y_t = \overline{Y}(Y_0 + 1 - e^{-t/t^*})$. Take $\overline{Y} = 5\%$, $Y_0 = 2\%$, and $t^* = 4$ years. Compute the present values of the caplets and add them to get the total value of the cap. Use the Black Scholes formula from earlier spreadsheets. Plan your Excel so that you can price one caplet then copy to get the other 11.

   (d) If you did part (c) wisely, it should take no more than ten seconds to replace the cap rate 7% with 6.5% and get the new cap value. Do the replacement, report the new value and the time it took to do the computation.

2. The *risk neutral implied probabilities* of a set of outcomes are the probabilities of those outcomes so that the market prices of the outcomes are their expected payouts. Suppose there are two coins, called X and Y, that take only two possible values S (for *solvent*) or D (for *default*). Let $p_X = \Pr(X = S)$, $p_Y = \Pr(Y = S)$, and $p_{XY} = \Pr(X = S \text{ and } Y = S)$. These are risk neutral probabilities, not actual probabilities. Let f_X be the price of the instrument that pays one unit if $X = S$ and zero if $X = D$. Let f_Y be the same for Y. Let f_{XY} be the price of the instrument that pays one unit only if $X = S$ and $Y = S$ (and pays nothing if either of X or Y is D). Suppose $f_X = .6$, $f_Y = .7$, and $f_{XY} = .5$. What are the risk neutral probabilities of the events $X = S$ and $Y = S$. What is the risk neutral implied correlation between X and Y? For the correlation, assume $S = 1$ and $D = 0$.

3. Suppose there are two bond ratings, `A` and `B`. Actual ratings agencies have more complex ratings. Take the following as the risk neutral default model. There is a constant intensity, λ, that an `A` rated bond will be *downgraded* to rating `B`. There is a constant intensity, μ, that a `B` rated bond will default. An `A` rated bond cannot default without being downgraded first. (Alas, top rated fixed income streams did default last year without being downgraded first.) Let $R(t)$ be the rating at time t, which can be either $R(t) = \texttt{A}$, or $R(t) = \texttt{B}$, or $R(t) = \texttt{D}$, the latter for default. Write $P_{\texttt{A}}(t) = \Pr(R(t) = \texttt{A})$ and the same for $P_{\texttt{B}}$ and $P_{\texttt{D}}$ We saw in class that starting from $R(0) = \texttt{A}$, $P_{\texttt{A}} = e^{-\lambda t}$.

   (a) Derive a differential equation for $P_{\texttt{A}}$ using the reasoning from class. That is

$$\begin{aligned} \lambda \delta t &= \Pr\,(\; R(t+\delta t) \neq \texttt{A} \mid R(t) = \texttt{A} \;)\;\;(+\;\textit{tiny}\;) \\ &= \frac{\Pr\,(\; R(t+\delta t) \neq \texttt{A} \text{ and } R(t) = \texttt{A} \;)}{\Pr\,(\; R(t) = \texttt{A} \;)}\quad(+\;\textit{tiny}\;)\;. \end{aligned}$$

   But $\Pr\,(\; R(t+\delta t) \neq \texttt{A} \text{ and } R(t) = \texttt{A} \;) = P_{\texttt{A}}(t) - P_{\texttt{A}}(t+\delta t)$, so you can rearrange and take $\delta t \to 0$ to get the differential equation for $P_{\texttt{A}}$. The solution of this differential equation is $P_{\texttt{A}}(t) = e^{-\lambda t}$, given that we know $P_{\texttt{A}}(0) = 1$ (why?).

   (b) Derive a differential equation for $P_{\texttt{D}}(t)$ in terms of $P_{\texttt{D}}(t)$ and $P_{\texttt{B}}(t)$.

   (c) Derive a differential equation for $P_{\texttt{B}}(t)$. This is the most complicated of the three, since it involves both $P_{\texttt{A}}(t)$ and $P_{\texttt{D}}(t)$.

   (d) Find the solution to the differential equation for $P_{\texttt{B}}(t)$ that satisfies the correct initial condition. You can use the ansatz $P_{\texttt{B}}(t) = ae^{-\lambda t} + be^{-\mu t}$. Assume $\lambda < \mu$. A low grade bond is more likely to default than a high grade bond is to be downgraded.

   (e) Find the solution to the differential equation for $P_{\texttt{D}}(t)$ from part (b) that satisfies the correct boundary condition. Hint: since there are three possible states, $P_{\mathtt{A}}(t)+P_{\mathtt{B}}(t)+P_{\mathtt{D}}(t) = 1$. You only have to check that the resulting formula for $P_{\mathtt{D}}(t)$ satisfies the differential equation and the correct initial condition.

(f) (not an action item) The numbers λ and μ are *transition rates.* The company *Credit Metrics* sells a table (matrix) of estimated transition rates for the full set of bond ratings. For example, there is a rate $\lambda_{\mathtt{Aaa}\to\mathtt{Aa}}$, and $\Pr\left(\; R(t+\delta t) = \mathtt{Aa} \mid R(t+\delta t) = \mathtt{Aaa} \;\right) \approx \lambda_{\mathtt{Aaa}\to\mathtt{Aa}}\delta t$. The probabilities of the various credit ratings at time t satisfy a system of differential equations involving these transition rates.

(g) Suppose a company has an outstanding zero coupon bond maturing in 5 years with a credit spread of 1% and a zero maturing in ten years with a 2% spread. What are the implied values of λ and μ? What should be the credit spread of a zero from that company maturing in twenty years? All bonds are rated `A` today. You may use the Excel equation solver, or trial and error, to solve the algebraic equations for λ and μ.