Assignment 2, due September 30

Corrections: (Sept 25, ?? became equation numbers. Sept. 28 problem 4 parts (d) and (e) were fixed to refer to $P(K)$ instead of $f(K)$.)

1. This exercise gives a slightly different way to determine the risk neutral probabilities in the binary and binomial tree models. The *risk neutral world* is a world in which future prices are truly random and follow the risk neutral probabilities. We do not live in the risk neutral world, but it helps to think about it. In the risk neutral world, the present price of any traded asset must be equal to the discounted expected value of its future price. Probabilities in the risk neutral world are called the *P measure.* Probabilities in the actual world are called the *Q measure.* Expectation using the risk neutral probabilities is written $E_P[\cdot]$. If A_0 is the known market price of a traded asset and A_T is the unknown price of that same asset "tomorrow", then

$$A_0 \;=\; CE_P\left[A_T\right] \;. \tag{1}$$

   (a) The value of cash today is 1. The value tomorrow is e^{rT}, in any state of the world tomorrow. Show that this means that the discount factor in (1) is $C = e^{-rT}$.

   (b) The price of a share of stock today is S_0 and the price tomorrow is S_T. Apply (1) to the asset price S. Assume that the possible values of S_T are uS_0 and dS_0. Show that this leads to formulas for the risk neutral probabilities $p_u = \Pr_{RN}\left(S_T = uS_0\right)$, and $p_d = \Pr_{RN}\left(S_T = dS_0\right)$.

   (c) It is particularly easy to find p_u and p_d if the future outcomes are symmetric with respect to the forward price $F_0 = S_0e^{rT}$, which means that $uS_0 - F_0 = F_0 - dS_0$. Why?

2. In a one period binary model, suppose that $B(0,T) = .95$, $u = 1.1$, $d = .9$, and that $S_0 = 100$. Go through the steps of finding the arbitrage price of a put option with strike price equal to the forward price of the stock. This involves finding numerical values for p_u and p_d. Also determine the amount of stock and cash in the replicating portfolio.

3. Suppose instead that there are three possible prices tomorrow, uS_0, mS_0, and dS_0, with $u > m > d > 0$. Suppose also that there is a liquid exchange traded "call" option that pays one unit of currency if $S_T = uS_0$ and nothing otherwise. Let A_0 be the present price of this option.

(a) Under what conditions (values of u, m, d, A_0, e^{rT}) is this market complete?

(b) Under what conditions is this market arbitrage free?

(c) Explain the equations to be solved to determine the risk neutral probabilities if the market is complete and arbitrage free. If the equations are very complicated, you need not give the solutions explicitly.

4. Suppose S_T can take any positive value, and that its probability density is $f(s)$. Suppose that $f(s) \to 0$ rapidly as $s \to 0$ and $s \to \infty$. Let $P(K)$ be the market price of a put option expiring at time T with strike price K. Suppose the discount factor is e^{-rT}.

   (a) Show that $P(K) \to 0$ as $K \to 0$. Interpret this in view of the probability that the option will finish (expire) in the money.

   (b) Find an approximate formula for $P(K)$ that holds as $K \to \infty$. Interpret this formula by thinking of the relationship between a put and a forward contract is it is virtually certain that the put will be exercised.

   (c) Suppose that there are market prices $P(K)$ for all K (a hypothesis Peter Carr is fond of). Using the integral expression for $P(K)$ in terms of $f(s)$, derive the formula

   $$f(K) \;=\; e^{rT} P''(K) \; . \tag{2}$$

   This is the *market implied probability* distribution of S_T. If we had less data than $P(K)$ for all K, we would get less information about $f(s)$, but it still could be a lot.

   (d) Show that (2) is consistent with the convexity of $P(K)$.

   (e) Show that if $P(K)$ is convex and has the large and small K behavior of parts (a) and (b), then (2) gives an $f(s)$ that is a probability density.

5. Programming exercise: postponed until assignment 3.