**Derivative Securities**, Courant Institute, Fall 2009
http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html
**Always** check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

# Assignment 3, due October 7

**Corrections:** (none yet)

1. This question 1 is to be done by hand, possibly with a calculator. It is very important that you understand the workings of the binomial tree model, so this will be the simplest computation that illustrates the main features of binomial tree pricing and dynamic replication. Assume there are 4 discrete times, three stages of the tree, and four possible final prices. Take the present spot to be $S_0 = 100$. Use the other parameters $B = e^{-r\delta t} = .9$, $u = 1.2$, and $d = .8$. The four possible prices at the final time are $S_{T,0} = S_0 d^3$, $S_{T,1} = S_0 d^2 u$, $S_{T,2} = S_0 du^2$, and $S_{T,3} = S_0 u^3$.
   (a) Compute the risk neutral probabilities $p_u$ and $p_d$.
   (b) Assume the option is a vanilla European put with strike price $K = 100$ expiring at the final time, fill in the binomial tree with option prices. This involves six computations.
   (c) What is the binomial tree price of the option at the initial time?
   (d) Does this price make sense, given the relationship between the strike price and the forward price?
   (e) Compute the values of $\Delta$, the amount of stock to hold in the replicating portfolio, at each internal node of the tree. This is another six computations.
   (f) Suppose you start at the present time with an amount of cash equal to the computed option price, and that you want to construct and maintain the stock/cash dynamic replicating portfolio for the option. Suppose the stock price trajectory is $S_0 \to uS_0 \to duS_0 \to d^2uS_0$. Describe the transactions that you must do at the initial time and at the two intermediate times. Show that at each time along this path, the total value of your stock/cash replicating portfolio is equal to the option value from part b. This will be easy if you use the results from part e.
   (g) Repeat part f with the path $S_0 \to dS_0 \to d^2S_0 \to d^2uS_0$. Comment on the differences between the transactions in the two cases and the differences in the result.

2. Download the file `bintree.cpp`. It is a C++ program that is a partial implementation of a binomial tree. If you compile and run it, it should produce a file called `option.csv`.

(a) Download the file `option.csv` from the class web site and check that it is the same as the one the program produced for you. A file in `.csv` format ("csv" is for *comma separated values*) is meant to be read by Microsoft Excel. Check that this works for you.

(b) Read over the program. You will see that it is incomplete in several crucial places. Most importantly, the actual computation of $p_u$ and $p_d$ has been left out. Modify the program to do the correct $p_u$ and $p_d$ then run it with the correct parameters to compare with your hand calculations from question 1. You may be tempted to do these tasks in the opposite order. Please resist this temptation.

(c) Modify the program to use the somewhat more realistic parameters $B = .99$, $u = 1.02$, $d = .98$, $n = 20$. Use Excel to make a plot of $P(K)$ for a range of $K$ values that shows the important behavior of $P(K)$. You will have to choose the number of $K$ values to use and the range of values. The procedure will be to run the C++ program then read the `option.csv` file into Excel for plotting[1]. Check that $P(K)$ has the qualitative behavior for large and small $K$ predicted in assignment 2.

3. Let $Z \sim \mathcal{N}(0,1)$ be a standard normal (mean zero, variance 1) random variable. The *cumulative normal* function is

$$N(x) \;=\; \Pr\left(Z \leq x\right) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz \;. \tag{1}$$

(a) Suppose $Y \sim \mathcal{N}(\mu,\sigma^2)$. Find a formula for $\Pr(Y < a)$ in terms of the cumulative normal function $N(x)$ and the parameters $a$, $\mu$, and $\sigma$. There are at least two ways to do this. One is to write the probability density for $Y$, express the probability as an integral, and change variables to express the integral in the form (1). This will be useful for part c below. Another way is to express $Y$ in terms of a standard normal and the parameters $\mu$ and $\sigma$.

(b) Suppose $X$ is a random variable with $\mu = E[X]$ and $\sigma^2 = \mathrm{var}(X)$. Suppose that $R_n = \sum_{k=1}^{n} X_k$, where the $X_k$ are independent random variables with the same distribution as $X$ (i.i.d. *samples* of $X$). Assuming the central limit theorem, write an approximate formula for $\Pr(R_n < a)$ in terms of the cumulative normal. The formula will be accurate for large $n$ and for appropriate values of $a$. *Warning*: the formula is not likely to be accurate when $a$ is much smaller than $E[R_n]$. The Central Limit Theorem gets tail probabilities wrong.

(c) In probability, the *indicator function* (or *characteristic function*) of a set $A$ is the function that takes the value $I_A(x) = 1$ if $x \in A$

[1] In a more perfect world, one could call the C++ program directly from Excel and automate the whole process. Microsoft makes this extremely hard to do, but not impossible.

and $I_A(x) = 0$ if $x \notin A$. This allows us to write (here $f(x)$ is the probability density of $X$.)

$$\Pr(X \in A) \;=\; E\left[I_A(X)\right] \;=\; \int I_A(x) f(x)\, dx \;=\; \int_{x \in A} f(x)\, dx \; .$$

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Find a formula for

$$Q(K) \;=\; E\left[I_{X \leq K} e^X\right]$$

in terms of the cumulative normal function and the parameters. You can do this with a combination tricks. One is to write the integral in terms of the standard normal density and complete the square to handle the $e^X$. This problem is the mathematical core of the Black Scholes formula.