Derivative Securities, Courant Institute, Fall 2009

http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 4, due October 14

Corrections: (October 10, #1(d) had the last sentence removed.)

1. This will give a derivation of the Black Scholes formula for Δ of a put that is different from the direct one in Hull and Kohn's notes. It will use the approach we used in class, and is in the class notes.

   (a) Use the formula for the put price as a discounted expected value in the risk neutral measure to show that

   $$P(s,K,r,T) \;=\; e^{-rT} E_{RN}\left[(K - sM_T)_+\right] \;,$$

   where the rest of the formula is independent of s.

   (b) Show that the derivative with respect to s of $E_{RN}\left[(K - sM_T)_+\right]$ is $-E_{RN}\left[M_T I_{S_T<K}\right]$.

   (c) Do the integral as we did the one in class to find a formula for $\Delta = \partial_s P(s,K,r,T)$. The formula should agree with the formula in the book (warning: Hull has a form that makes this equivalence hard to see.).

   (d) The derivative of the cumulative normal is $N'(x)$, and is given by

   $$N'(x) \;=\; \partial_x \int_{-\infty}^{x} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \;=\; \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \;.$$

   Find a formula for $\partial_s d_1$ that allows you to go from the answer of part (c) to a formula for $\Gamma = \partial_s \Delta = \partial_s^2 P$.

2. Suppose S_t^n is a discrete time binomial tree process that evolves according to the usual rule that $S_{t+\delta t}^n = uS_t^n$ with probability p_u and $S_{t+\delta t}^n = dS_t^n$ with probability p_d. Use parameters $u = 1 + \sigma\sqrt{\delta t}$, $d = 1 - \sigma\sqrt{\delta t}$, and $p_u = p_d = \frac{1}{2}$. Let $X_t^n = \ln(S_t^n)$ be the log price process.

   (a) Show that for each n, the values $X_{t_k}^n$ for an ordinary arithmetic random walk.

   (b) Suppose that $T = n\delta t$ and that $n \to \infty$ as $\delta t \to 0$ with T fixed. Use Taylor series and the Central Limit Theorem to find the distribution of X_T^n in the limit.

   (c) Use this to show that in the same limit S_T^n becomes lognormal.

(d) Let S_T and X_T be random variables with the appropriate limiting distributions. Show that $E[S_T] = S_0$ but that $E[X_T] \neq X_0$.

3. Over this and a few coming assignments, you will develop option pricing functions in Excel. Over time you will learn more features of Excel (if you're a beginner) and more about options. This week it will just be the Black Scholes price for a put.

   (a) Download the spreadsheet `a4.xls` and check row 6 to make sure that the formulas are right. Delete all rows below row 6. Add formulas for Δ and Γ from question 1 above. The extra items in columns F, G, and H are put to the side, and would be completely hidden on a professional version of this.

   (b) Check that your formulas for Δ and Γ are correct by using finite approximations to the derivative. That is, $\Delta(s) \approx \left(P(s+\delta s) - P(s)\right)/\delta s$, and a similar finite difference approximation for Γ. Do this with a decreasing sequence of δs values to see that the finite difference approximation is converging to the right thing as $\delta s \to 0$. Warning: numerical evaluation in Excel often has unnecessarily large errors that will make this stop working when δs is too small.

   (c) (Writing assignment) For $S_0 = 100$, experiment with different parameter values (r, σ, and T). Describe in a few sentences what happens to option prices and Greeks. Give rough intuitive explanations in a few sentences.

   (d) Make plots of the put price and Greeks as a function of S_0 with the other parameters fixed. Do this for the range $20 \leq S_0 \leq 150$.

   (e) Check in the plot that P and Δ have the same small S_0 behavior of a forward contract with the same strike and expiration.

4. Over this and a few coming assignments, you will develop your C++ option pricer. This week, it will be to calibrate the tree to given market parameters and to validate using the Black Scholes formula.

   (a) Download the shell code `putTree.cpp`. It is an edited version of the code from Assignment 3, but with the calling arguments changed and the guts removed. Do not change the definitions of `dt`, `u`, `d`, or `B`, but fill in the rest so that it does a correct binomial tree calculation with these parameters.

   (b) Do the run asked for and look at the output file with Excel. You will see that the parameters K, r, etc. are supplied to the spreadsheet. This means that you can look at the spreadsheet and know exactly what run it was, with parameters given. Modify the program so that it also includes the parameter `n` in the output. It should go after `T` in the second row. Also a label `n` in the first row above its value in the second row, as was done with the other parameters. In a perfect world, the C++ program would get its parameters directly from a spreadsheet. But this is very hard to do in practice.

(c) We argued in class that as $n \to \infty$, the results of this code should converge to the values given by the Black Scholes formula. Do a *convergence study* to verify that this is true. That is, run the C++ program with an increasing sequence of n values (such as 25, 50, 100, 200, 400, 800, as far as needed) to see that the tree code values of P and Δ converges to the Black Scholes formula values. So this for the range of S values in the code as it was posted.