

## Assignment 5, due October 21

**Corrections:** (Oct 19: Question 3(a), changed  $a(T_t)$  to the correct  $a(Y_t)$ .)

1. We spoke in class about “small” and “tiny” terms in approximations to a stochastic (Ito) integral. The definition of a tiny term was that when you add up all the tiny terms, even the sum goes to zero as  $n \rightarrow \infty$ . The rule was that a term  $G_k$  is tiny if  $|G_k| = O(\delta t^p)$  with  $p > 1$ , or if  $|G_k| = O(\delta t)$  and  $E[G_k] = 0$ . This is not exact mathematics, but it makes a reasonable rule of thumb. This exercise asks you to verify that the rule is correct in the following setting. Recall that a random function  $F(t)$  is *adapted* if the value of  $F(t)$  is determined by the information  $\mathcal{F}_t$  available at time  $t$ . We also assume that the Brownian motion path  $W$  has the property that increments  $W_s - W_t$  for  $s > t$  are completely independent of any information available at time  $t$  (i.e. in  $\mathcal{F}_t$ ).

- (a) Evaluate  $E[Z^4]$  when  $Z$  is a standard (i.e. mean zero, variance one) normal. Hint: write the integral as  $C \int z^3 \left( z e^{-z^2/2} \right) dz$ . You can write  $z e^{-z^2/2} = -\partial_z e^{-z^2/2}$  and integrate by parts.
- (b) Evaluate  $E[X^4]$  when  $X \sim \mathcal{N}(0, \sigma^2)$ . Hint: You do not have to integrate again. Instead, you can reduce this to part (a) by scaling, or by representing  $X$  in terms of a standard normal.
- (c) If  $W$  is standard Brownian motion, evaluate  $E \left[ (W_t^2 - t)^2 \right]$ . Hint: you do not need to do any integrals.
- (d) Suppose  $W$  is a standard Brownian motion and that  $t_1 < t_2 < t_3 < t_4$ . Evaluate

$$E \left[ \left( \left\{ (W_{t_2} - W_{t_1})^2 - (t_2 - t_1) \right\} + \left\{ (W_{t_4} - W_{t_3})^2 - (t_4 - t_3) \right\} \right)^2 \right].$$

Hint: The square terms are done as in part (c). You can figure out the cross term using the independent increments property. Do not expand what's in the braces:  $\{\cdot\}$ .

- (e) Suppose  $F(t)$  is an adapted random function of  $t$  and consider the sum of tiny terms

$$R = \sum_{t_k < T} F(t_k) (\delta W_k^2 - \delta t) .$$

Here, we use the usual conventions:  $t_k = k\delta t$ ,  $\delta t = T/n$ , and  $\delta W_k = W_{t_{k+1}} - W_{t_k}$ . Show that  $E[R] = 0$ . Assume that  $E[F(t)^2] \leq A$  for all  $t \leq T$ . Show that  $E[R^2] = \text{var}(R) \leq 2AT\delta t$ . Hint: evaluate the cross terms as in part (d), again using even more strongly the independent increments property, as it relates to values of  $F$  and increments of  $W$ .

(f) What does part (e) say about the size of  $R$  as  $\delta t \rightarrow 0$ ?

2. To prepare for question 3, this question walks you through the *Ito isometry formula*. Consider the Ito integral  $I = \int_0^T F(t)dW_t$  and the approximations  $I_n = \sum_{t_k < T} F(t_k)\delta W_k$ , with notation as in question 1.

(a) Use the reasoning of question 1, part (e) to find a formula for  $E[I_n^2]$

(b) Take the limit  $\delta t \rightarrow 0$  as usual to derive the formula

$$E \left[ \left( \int_0^T F(t)dW_t \right)^2 \right] = \int_0^T E[F(t)^2] dt .$$

Note that the integral on the left is an Ito integral, while the integral on the right is a Riemann integral that has nothing to do with Brownian motion.

(c) (Not an action item) This formula is very helpful in proofs involving the Ito integral.

3. Suppose the risk neutral process is a geometric Brownian motion with a deterministic *term structure of volatility*

$$dS_t = rS_t dt + \sigma_t S_t dW_t . \tag{1}$$

*Term structure* means that  $\sigma_t$  is a function of  $t$ . *Deterministic* means that the function  $\sigma_t$  is known at time 0. Suppose that  $r$  is a known constant.

(a) Use reasoning similar to that we used in class to argue that if  $dY_t = a(Y_t)dt + b(Y_t)dW_t$ , and if  $u(y)$  is a smooth function of  $y$ , then  $du(Y_t) = u_y(Y_t)dY_t + \frac{1}{2}u_{yy}(Y_t)b^2(Y_t)dt$ . This is a more general form of Ito's lemma.

(b) Let  $X_t = \ln(S_t)$  be the log process. Use the more general form of Ito's lemma (even if you were unable to verify it) to find the SDE that  $X_t$  satisfies.

(c) Show that  $X_T$  is Gaussian if  $\sigma_t$  is deterministic. You can get this conclusion by looking at finite sum approximations to the Ito integral  $\int_0^T \sigma_t dW_t$ .

(d) Use the Ito isometry formula to find the variance of  $X_T$ .

(e) Find a formula like the Black Scholes formula for the price of a European call option with strike  $K$  expiring at  $T$ . Your formula will involve  $V_T = \int_0^T \sigma_t^2 dt$ .

4. A vanilla *American style* option on underlier with payout  $V(s)$  may be exercised at any time up to and including  $T$ . Exercising at time  $t$  gives the holder payout  $V(S_t)$  at time  $t$ . The solution has a simple structure, which I explain here for a put. There is a *critical price* (also called *early exercise price*),  $S_*(t)$ . When  $S_t \leq S_*(t)$ , the holder should exercise the option. When  $S_t > S_*(t)$ , the holder should not exercise. We can determine the early exercise price (approximately) as part of the binomial tree calculation below. If  $s \leq S_*(t)$ ,  $f(s, t) = V(s)$  (why?).

There is no efficient formula for pricing American style options in the binomial tree or geometric Brownian motion models. Instead, we must use numerical methods. This exercise leads to a binomial tree algorithm for pricing American style options. Let  $f(s, t)$  be the price of the option at time  $t$  if  $S_t = s$ . In the binomial tree model,  $t$  and  $s$  takes only finitely many values. The binomial model will be that in one step of size  $\delta t$ , the stock may go up by a factor of  $u = 1 + \sigma\sqrt{\delta t}$  or down by a factor of  $d = 1 - \sigma\sqrt{\delta t}$ . In addition, the holder may decide at time  $t$  to exercise the option, which means to receive payment  $V(S_t)$ . In general, the holder will choose to exercise the option if  $V(S_t)$  is larger than the expected discounted value that comes from not exercising (holding) at time  $t$ . Of course, in the risk neutral world, this expected discounted value also is the option price.

The binomial tree algorithm for pricing American style options is at each stage to compare compute the value of the option from continuation to the value from exercising, and take the option value to be the larger of these two. The value from continuation is the discounted risk neutral expected value from the two possible future states at time  $t + \delta t$ .

- (a) Modify the `putTree.cpp` code from Homework 4 to price American style vanilla put options. Use parameters from that assignment:  $K = 100$ ,  $\sigma = .2$ , and  $T = .5$ , but take  $r = .06$  (higher than today). Compute the put price for the same range of spot prices  $S_0 \in [20, 150]$ .
- (b) Make a plot of the prices with American and European prices as a function of  $S_0$  (use Excel as before). You can get the European prices with the Black Scholes formula in your spreadsheet. Plot the *American premium*, the difference between American and European prices.
- (c) Make a plot of  $\Delta$  as a function of the spot in the same range. Note that  $\Delta = -1$  below the early exercise price. Does  $\Delta$  approach  $-1$  in a continuous fashion as the spot goes down to the early exercise price?
- (d) (Writing assignment) Experiment with other parameter values and determine when the American feature adds the most relative value to the option price. Try to give an intuitive explanation.