Assignment 6, due October 28

Corrections: (none yet)

1. Suppose that the government were to charge a tax on stock ownership that would require the owner to pay mdt per share if she owned the share during the interval $(t, t + dt)$. Assume m is a known constant. Assume that this applies linearly even for fractional or short positions (in which case she receives money). Modify the continuous time hedging argument we gave in class to find the Black Scholes PDE that would price European options. What is the risk neutral process so that the solution to this PDE is the value function $f(s, t) = e^{-r(T-t)}E_{s,t}[V(S_T)]$? Warning: this is not the cost of carry model, though it is related.

2. This exercise will construct a number of solutions to the Black Scholes equation.

   (a) Suppose a European style *power contract* pays S_T^p at time T. Show that there is a solution to the corresponding Black Scholes equation of the form $f(s, t) = A(t)s^p$. Find an explicit formula for $A(t)$. I have not heard of actual power contracts in the marketplace, but maybe there have been some.

   (b) Find the solution to the Black Scholes equation corresponding to the *log contract* that pays $\ln(S_T)$ at time T. Hint: look at what happens when you apply $s^2\partial_s^2$ and $s\partial_s$ to $\ln(s)$ and try to guess what you have to include to get a full solution of the Black Scholes equation that satisfies the final condition.

   (c) Find the solution for the simple *digital* contract that pays $V(S_T) = 1$ if $S_T < K$ and $V(S_T) = 0$ if $S_T \geq K$. Hint: use the probability representation of the solution $f(s, t) = e^{-r(T-t)}E_{s,t}[V(S_T)]$. The answer will involve the cumulative normal $N(d_1)$ (or maybe $N(-d_2)$ or something like that). Verify that your solution satisfies the Black Scholes PDE by explicitly calculating the derivatives involved. Verify that your solution satisfies the appropriate final conditions.

3. This exercise explores the log variable transformation in the Black Scholes one more time. The log variable simplifies the PDE as well as the stochastic process. It allows us to understand the instability of trying to run the Black Scholes PDE in the wrong direction.

(a) Compute the Black Scholes Equation in the variable $x = \ln(s)$. That is (in the clumsy way mathematicians say such things), let $g(x, t)$ be defined from $f(s, t)$ by $g(\ln(s), t) = f(s, t)$. Suppose that f satisfies the Black Scholes PDE. Calculate the PDE satisfied by $g(x, t)$. You can do this using the chain rule, such as $\partial_s f = \partial_x f \cdot \frac{dx}{ds} = \frac{1}{s} \partial_x f$, etc.

(b) Show that the random process corresponding this PDE is the log process we found before using Ito's lemma.

(c) Suppose the g equation has final condition $g(x, T) = e^{ikx}$ where k is a real number (called the *wave number*), and $i = \sqrt{-1}$. Show that the solution may be written $g(x, t) = A(t)e^{ikx + i\omega t}$, where ω and A are both real and depend on k and t and T, but not on x. Find the explicit formula for A. Describe the size of $A(0)$ when k is large. Is it exponentially large or small?

(d) Consider the setup of part (c) but with *initial conditions* given $g(x, 0) = e^{ikx}$. Find an explicit formula for $|g(x, T)|$ (which is independent of x) and state whether this is exponentially large or exponentially small as $k \to \infty$.

(e) Discuss the significance of the answers to parts (c) and (d) in terms of the relationship between the the size of the solution as a function of the size of the final or initial data.

4. This question asks you to compute Black Scholes implied volatility for calls and puts on the S&P500 index, which is called `SPX`. The goal is to understand implied vol, and to observe skew and smile (which together form "smirk"), in option price data. Download the Excel spreadsheet `a6.xls` from the homework web page. This has a VBA macro[1] called `EPutIVol` that computes the Black Scholes implied volatility for a European put. You will have to "enable macros" when you open `a6.xls` in Excel, but I promise that I did not put a virus in it. To see the VBA code, in Excel, click on "Tools", then "Macro", then "Visual Basic Editor", then in the "Projects" window open the "Modules" folder (if it isn't already open) and doubleclick on "Module 1". If everything is working correctly – *please post ASAP on the class message board otherwise* – the price in cell `C9` should be (almost) the same as the price in cell `B6`. The sheet is programmed to compute the implied vol for given put parameters and put the result in cell `C8`. It then computes the price for the vol in cell `C8`, which goes into cell `C9`. Try changing the put price `B6` to see that the implied vol calculator is working correctly.

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[1]It would be natural to do this using the Excel `solver`, which solves nonlinear equations. Unfortunately, I was unable to do this in a way that spreadsheet cells would automatically update themselves as they should in Excel. Everything I tried left me having to use the solver dialog box every time a data number changed. I would be very happy if someone could show me how to avoid going through the dialog box. It seems inconceivable that Microsoft Corporation would place this pointless obstruction to automaton in front of one of their more useful functions, particularly since it drives people to non-Microsoft workarounds.

(a) Create a new macro called `ECall` that computes the Black Scholes value for a European call. It should be easy to copy and modify the code for `EPut`. You only have to change the next to last line. The clunky statement `WorksheetFunction.NormSDist(d)` in VBA produces the same result as `NormSDist(d)` in Excel, and the same for many other Excel built in functions (but not `Exp`, for example). Call your `ECall` from `a6.xls` and check that the result is correct.

(b) Copy and modify the VBA code for `EPutIVol` to make a macro `ECallIVol` that computes implied vol for a European Call. Test that it produces the right answer in the way the put implied vol code is tested in the posted `a6.xls`.

(c) The put implied vol calculator relies on the true assumption that the price is an increasing function of the vol. How would the blocks of code starting with `If V1 < V Then` and `If V1 > V Then` need to change if V were a decreasing function of σ?

(d) The following information comes from the CBOE web page `http://www.cboe.com/delayedquote/QuoteTable.aspx` on October 19 at 12:36 pm. At that time the S&P 500 index (SPX) was 1097.55. European puts on SPX expiring in November ("the Saturday after the third Friday in November") had the following prices

Strike	Bid	Ask
1090	21.5	22.7
1095	23.6	24.8
1100	25.9	27.2
1105	28.4	29.7

Puts

Strike	Bid	Ask
1090	27.7	28.8
1095	24.6	25.7
1100	22.2	22.5
1105	19.5	20.5

Calls

Put these numbers into a spreadsheet and compute and plot the implied vols. For the option price, use the midpoint between bid and ask. Does the graph have the skew and smile? For time to expiration, use $T = .1$years. This is not exact (we will talk about day counting later), but the error does not change the shape of the implied vol graphs (why not?). Use risk free rate $r = .02$, which is too large, but does not matter.

(e) Do these put and call prices satisfy put/call parity? If not, can you suggest a trade that is guaranteed to make money?