Assignment 7, due November 4

Corrections: (Oct. 31: fixed equation (2) to have dy instead of dx.)

1. Consider the backward equation in the log variable from Assignment 6, but without discounting:

$$0 \;=\; \partial_t f \;+\; \frac{\sigma^2}{2}\partial_x^2 f \;+\; \left(r - \frac{\sigma^2}{2}\right)\partial_x f \;. \tag{1}$$

We may write the solution as

$$f(x,t) \;=\; E_{x,t}\left[V(X_T)\right] \;.$$

(a) Describe the random variable X_T under the condition that $X_t = x$ (with $t \leq T$). What kind of distribution does it have? What are the parameters? Let $u(y,x,T-t)$ be the probability density of X_T (in the variable y) under the condition $X_t = x$.

(b) Use the answer to part (a) to write an explicit formula for $f(x,t)$ of the form

$$f(x,t) \;=\; \int_{-\infty}^{\infty} V(y)u(y,x,T-t)\,dy \;. \tag{2}$$

(c) Verify by explicit differentiation with respect to t and x that the function defined by (2) satisfies the PDE (1). Note that all the derivatives of the integral fall onto the probability density u rather than on the payout function V. The formula (2) would be called the *Green's function* representation of the solution of the final value problem (1) with final condition $f(x,T) = V(x)$. The *Green's function* (also called *fundamental solution*) is $u(y,x,t)$.

(d) Find a explicit formulas for the quantities

$$M_1(t) \;=\; \int_{-\infty}^{\infty} \left|\partial_y u(y,x,t)\right|\,dy \;\;, \qquad M_2(t) \;=\; \int_{-\infty}^{\infty} \left|\partial_y^2 u(y,x,t)\right|\,dy \;.$$

Hint 1: Start by finding *scaling relations* such as $M_1(t) = t^p M_1(1)$, and of course, that M_1 and M_2 are independent of x. Find p. Hint 2: Then show that

$$\int_{-\infty}^{\infty} \left|\partial_y u(y,0,1)\right|\,dy \;=\; 2\max_y \left|u(y,0,1)\right| \;.$$

and evaluate the maximum. For M_2, the factor of 2 in the right must be replaced by a different factor.

(e) Use the calculations in part(d) and the representation (2) to prove (somewhat informally) the inequalities

$$|\partial_x f(x,t)| \;\leq\; \frac{C_1}{\sqrt{T-t}} \max_y |V(y)|$$

$$\left|\partial_x^2 f(x,t)\right| \;\leq\; \frac{C_2}{T-t} \max_y |V(y)| \;.$$

What formulas for the constants C_1 and C_2 do you get from part (d)?

(f) Show (somewhat informally) that the value function f is at least twice differentiable in x even if the payout function is discontinuous. This is the *smoothing* property of the backward equation (1). It makes even rough payout functions into smooth value functions.

(g) The *butterfly* value function is[1] $b(x) = (1-|x-K|)_+$. Make a graph of $b(x)$. Is there a bounded payout function $V(y)$ so that the value function satisfies $f(x,t) = b(x)$ for some $t < T$? What does this say about the possibility of solving the backward equation forward in time with initial condition $f(x,0) = b(x)$?

2. The Black Scholes equation with discounting in the log variable is $\partial_t f + \ldots - rf = 0$. Show that this may be reduced to the backward equation without discounting (1) by multiplication by e^{-rt}. Show in a similar way that solutions of the equation $\partial_t f + \frac{\sigma^2}{2}\partial_x^2 f + a\partial_x f = 0$ may be reduced to the same equation with $a = 0$ by multiplication by e^{bx} and then by e^{ct} for suitable b and c. Check that this by applying it to the function $u(y,x,t)$ of question 1. Show that you transform u to a solution of the PDE (1) without the $\left(r - \frac{\sigma^2}{2}\right)\partial_x f$ term.

3. The *perpetual put* is the limit of the value function for an American style put as the expiration time goes to infinity. Let the value of the put at time t with expiration T be denoted by $f(s,t,T)$. The answer to this question may be found quickly with a search engine. Please do not do this.

(a) Show that $f(s,t,T)$ is an increasing function of T, for any fixed value of s and t.

(b) Show that $f(s,t,T)$ does not go to infinity as $T \to \infty$ by finding a simple number A with $f(s,t,T) \leq A$ for all s,t,T.

(c) Argue by picture or by mathematical theorem that for each s and t, the limit $\lim_{T\to\infty} f(s,t,T) = g(s)$ exists, and is independent of t.

(d) Suppose that $g(s) > V(s)$ for $s > b$. Find the differential equation g satisfies for $s > b$ and say what its solutions are.

(e) For what values of r and σ may these functions be pasted smoothly to $g(s) = V(s)$ for $s < b$ in a way that defines $g(s)$ for all $s > 0$?

[1] OK, it is supposed to be in the s variable rather than x, but the point is the same.

4. Does put/call parity apply to American style options? Make a plot (using Excel) to illustrate your answer using computed American option prices. Does put/call parity apply to European style options in the presence of a non-zero cost of carry, q?

5. A *Bermudan option* is a put or call option that may be exercised on any of a small number of specified exercise dates. Suppose the exercise dates are $t_1 < t_t < \cdots < t_m = T$. Describe how the binomial tree algorithms we have used to value European or American style options would have to be modified to value a Bermudan option.

6. A digital option pays $V(s) = 1$ if $s > K$ and $V(s) = 0$ if $s < K$. Describe the early exercise strategy for an American style digital option. What PDE and final conditions and boundary conditions would you solve to compute the price?