

Derivative Securities, Courant Institute, Fall 2009

<http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html>

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 8, due November 11

Corrections: (**Nov. 6**, corrected #1b to replace σ by s in the hint. **Nov. 9**, fixed parts of #2(b): the coefficients changed from $\sigma_0 = .3$ to $\sigma_0 = .2$, and $\sigma_1 = .01$ to $\sigma_1 = -.01$. The formula for dS_t had $\sigma(S_t)dW_t$ changed to its present form of $\sigma(S_t)S_t dW_t$.)

1. We are going to calculate some solutions to the backward equation

$$\partial_t f + \frac{\sigma^2}{2} \partial_x^2 f + a \partial_x f = 0. \quad (1)$$

If the payout is $V(x)$, then the undiscounted value function is

$$f(x, t) = E_{x,t} [V(X_T)]. \quad (2)$$

Here, of course, $dX = \sigma dW + a dt$ and $X_t = x$. Some of the material in this question is repeated from previous assignments, but I want to have it all collected here.

- (a) What is the distribution of X_T ?
 - (b) Suppose $V(x) = x^4$. Use (2) to find a formula for $f(x, t)$. Hint: if $X \sim \mathcal{N}(\mu, s^2)$, then $X \sim \mu + sZ$ (the left and right sides have the same distribution) with $Z \sim \mathcal{N}(0, 1)$. Therefore $E[X^4] = \mu^4 + 6\mu^2 s^2 + 3s^4$.
 - (c) Show by direct calculation that the function $f(x, t)$ from part (b) satisfies the backward equation and has the right final values.
 - (d) (extra credit) Calculate $f(x, t)$ for final condition $V(x) = e^{-x^2/2k^2}$. Hint: write the expectation (2) as an integral with respect to the standard normal as in part (b). This leads to an integral with a quadratic in the exponent. You can calculate this integral by completing the square in the exponent. When k is small the payout is very small unless X_T is very close to zero. Make some rough sketches of the corresponding value function to see where the value function is large or small. When σ is also small you can see the role of a as drift more clearly. Check by explicit differentiation that your formula for f satisfies (1).
2. Download the files `randn.cpp`, `path.cpp`, and `randMain.cpp`. They contain C++ code that generates sample paths of geometric Brownian motion using the forward Euler method discussed in class.

- (a) Verify that the codes work by modifying them to compute the value of a European put option with parameters $\sigma = .3$, $r = .03$, $K = 100$, $S_0 = 100$, and $T = 1$ (Use the Excel functions you already wrote for this purpose.). Experiment with the computational parameters – the number of time steps and the number of sample paths – to see what leads to accurate results. Do not forget to put in the overall discount factor.
- (b) Consider a volatility skew and smile model of the form

$$\sigma(s) = \sigma_0 + \sigma_1(s - s_0) + \sigma_2(s - s_0)^2 . \quad (3)$$

Use the values $s_0 = 100$, $\sigma_0 = .2$, $\sigma_1 = -.01$, and $\sigma_2 = 2 \cdot 10^{-4}$. The risk neutral process is $dS_t = rS_t dt + \sigma(S_t)S_t dW_t$. Compute the values of European put options with a range of strikes from $K = 70$ up to $K = 130$. Experiment with the computational parameters until you are confident that the results are accurate. It may be that the program takes more than a few seconds to run – Monte Carlo is slow.

- (c) Import the results of part (b) into Excel and compute the Black Scholes implied volatilities using the plugin from an earlier assignment. Is the implied volatility function equal (3)? If so, why? If not, why not? In particular, if the implied vol is higher or lower than $\sigma(s)$, explain why is higher or lower, as the case may be. Do this as best you can – it will be hard to find a rigorous explanations.
3. The forward Euler solution of the backward equation for question 2 is postponed until next week.