Assignment 9, due November 18

Corrections: (none yet)

1. Consider the stochastic integral $M_t = \int_0^t b_u dW_u$.
   (a) We made a general argument that the Ito differential of any quantity Q_t should be defined by what you have to integrate (an Ito integral plus a Riemann integral) to get $Q_t - Q_0$. Apply this reasoning to find a formula for dM_t.
   (b) Apply the reasoning from Ito's Lemma (small, tiny, etc.) to get a formula for de^{M_t}.
   (c) Suppose σ_t is a random but adapted vol parameter. For example, there could be a function $\sigma_t = \sigma(S_t)$ as in assignment 8, or σ_t could evolve according to a separate random process, as in the Heston model (see Continuous Time Finance for details). Show that the solution of $dS_t = rS_t dt + \sigma_t S_t dW_t$ is given by
   $$S_t = S_0 \exp\left(\int_0^t \sigma_u dW_u + \text{a Riemann integral} \right) . \qquad (1)$$
   Find the Riemann integral and complete the formula (1).
   (d) A different way to do this is to write an integral expression for the log process $X_t = \log(S_t)$ and exponentiate at the end. Show that this gives the same formula (1).

2. Suppose $dS = \mu S dt + \sigma S dW$ as usual. Suppose there is a *leveraged* version of S, called L_t, that satisfies $\frac{dL_t}{L_t} = m \frac{dS_t}{S_t}$. Show that this is not the same as $L_t/L_0 = mS_t/S_0$ by finding the correct formula for L_t in terms of S_t. (This comes from the research of Marco Avellaneda and Stanley Zhang. There are actually traded leveraged versions of the S&P 500 (called QQQQ) with $m = 3, 2, -1, -2$.)

3. Assume that a bond priced at par has an annual 6% coupon and matures in 30 years. All the following questions can be answered with paper and pencil algebra using formulas like
   $$\sum_{k=m}^{n} z^k = \frac{z^m - z^{n+1}}{1-z} \quad , \quad \sum_{k=m}^{n} k z^k = z \frac{d}{dz} \frac{z^m - z^{n+1}}{1-z} \quad .$$
   But you can use Excel to do the sums if the algebra takes too long.

(a) What is its yield to maturity if interest is compounded annually (easy question!)?

(b) What is its yield to maturity if interest is compounded continuously?

(c) What is its duration? Does this depend on how interest is compounded?

(d) How much does the duration increase if the maturity extends to 100 years?

4. Look at the data about US Treasury yields. First, results of recent auctions from the web page

http://www.treasurydirect.gov/RI/OFNtebnd

Recent Note, Bond, and TIPS Auction Results

Security	Term	Type	Issue Date	Maturity Date	Interest Rate %	Yield %	Price Per $100	CUSIP
3-YEAR		NOTE	11-16-2009	11-15-2012	1.375	1.404	99.915148	912828LX6
10-YEAR		NOTE	11-16-2009	11-15-2019	3.375	3.470	99.203098	912828LY4
30-YEAR		BOND	11-16-2009	11-15-2039	4.375	4.469	98.454984	912810QD3
2-YEAR		NOTE	11-02-2009	10-31-2011	1.000	1.020	99.960585	912828LT5
5-YEAR		NOTE	11-02-2009	10-31-2014	2.375	2.388	99.938982	912828LS7
7-YEAR		NOTE	11-02-2009	10-31-2016	3.125	3.141	99.899963	912828LU2
4-YEAR	6-MONTH	TIPS	10-30-2009	04-15-2014	1.250	0.769	104.115658	912828KM1

Then the Treasury yield curve computed by the Fed as posted on their site

http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yield.shtml

Daily Treasury Yield Curve Rates

Get e-mail updates when this information changes.

Historical

XML This data is also available in XML format by clicking on the XML icon

November 2009

Date	1 mo	3 mo	6 mo	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
11/02/09	0.03	0.06	0.17	0.38	0.92	1.44	2.33	3.00	3.45	4.22	4.26
11/03/09	0.04	0.06	0.17	0.38	0.92	1.46	2.36	3.05	3.50	4.29	4.34
11/04/09	0.05	0.05	0.16	0.36	0.91	1.46	2.39	3.09	3.57	4.36	4.41
11/05/09	0.06	0.04	0.16	0.36	0.90	1.44	2.35	3.06	3.57	4.36	4.41
11/06/09	0.06	0.06	0.16	0.34	0.86	1.40	2.30	3.02	3.54	4.35	4.40

Note that treasury notes (intermediate duration) and bonds (long duration) pay dividends semi-annally. The number in the *Interest Rate %* column specifies the size of the coupon payments. For example, if the Interest Rate % is 6%, that means there are two $3 coupon payments per year per $100 of bond.

(a) Suppose we want to build a partial yield curve using the treasuries auctioned on 11/2/09. It would be tempting for a mathematician to use the formulas of Question 3 above by assuming the yield is constant for the first two years, a different constant between years 2 and 5, and a different constant beyond year 5. Does this assumption match the Treasury yield curve data below? If not, where is the biggest discrepancy?

(b) The yield (which is yield to maturity) for the 2 year Note is larger than the yield curve yield for two years on the same day (11/2/09). Why?

5. Build a PDE based pricer for European and American style options with fixed volatility functions $\sigma(s)$ as in Assignment 8. Modify the C++ program from earlier assignments that does the binomial tree algorithm to solve the backward equation for the log price process $dX_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW$. Use the forward Euler method discussed in class. Modify the program so that instead of computing a tree, it computes on a fixed interval $x_{\min} \leq x \leq x_{\max}$. The values of $f(x_{\min}, t)$ and $f(x_{\min}, t)$ should be specified to the program. Of course, the final conditions also need to be specified. Also specified should be the computational parameters `NX` and `NT`, the number of x points and t points respectively. This means that $\delta x = (x_{\max} - x_{\min})/NX$ and $\delta t = T/NT$. The x points are $x_k = x_{\min} + k\delta x$ and $t_j = j\delta t$.

   (a) For a European put, the boundary conditions should be that when x is small the put price is equal to the corresponding forward price, and when x is large the put price is equal to zero. See what values of $x_{\min}$ and $x_{\max}$ you have to use to compute the price of the put from assignment 8 with $K = 110$ with an error less than .01. You will also have to take large values of the computational parameters NX and NT. Make sure that NT satisfies the CFL stability restriction.

   (b) Explain how you code would be modified to price an American style put option with the same parameters. You need not make these modifications or hand in results from them.

   (c) Explain how you would modify the program to compute the put price with the quadratic volatility function $\sigma(s)$ from assignment 8. The issues are: (i) How do you compute the values $f_{k,j}$ from the values $f_{k,j+1}$? (ii) What stability restriction do you need? We will do this in Assignment 10.