Derivative Securities

Class 2
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Lecture outline

latest correction: none yet

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http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html

Outline

Arbitrage, pricing, risk neutral probabilities

- General abstract discrete model
- Definition of arbitrage
- The geometry
- *"No arbitrage"* is equivalent to "*there exist risk neutral probabilities"*
- Complete market -- a new instrument can be replicated
- The one period binomial model, the Δ hedge
- The multi-period binomial model, the binomial tree
- Rebalancing and dynamic replication

General abstract discrete model

- N instruments, $i = 1,\ldots,N$
- C_i = price today of instrument i
- Prices may be positive or negative
- M possible states of the world "tomorrow", $j = 1,\ldots,M$
- V_{ij} = price tomorrow of instrument i in state j
- Π = portfolio purchased today
- W_i = weight of instrument i in Π
- Weights may be positive or negative
- Cost/value of Π today is $\Pi_0 = \Sigma_{i=1}^{N} W_i C_i$
- Cost/value of Π in state j tomorrow is $\Pi_{T,j} = \Sigma_{i=1}^{N} W_i V_{ij}$

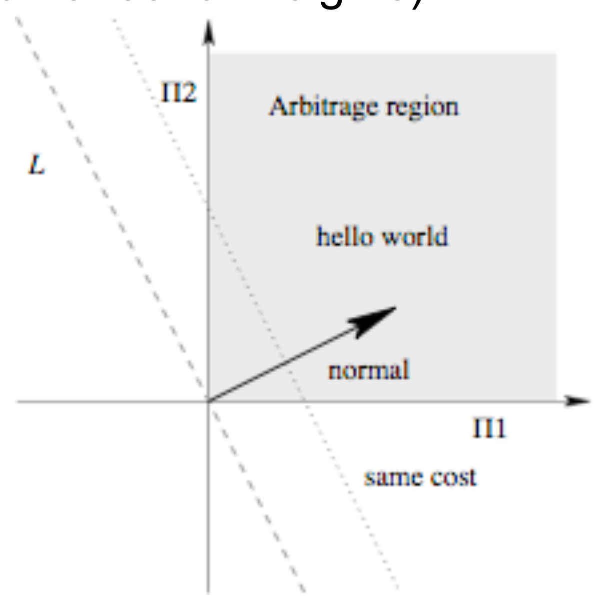
Π is an *abstract arbitrage* if:

- $\Pi_0 = 0$
- $\Pi_{T,j} \geq 0$ for all j
- $\Pi_{T,j} > 0$ for some j

Axiom: the model is *arbitrage free* --
no such Π exists

Geometry and linear algebra

- Cash flow vector: $\Pi_T = (\Pi_{T,1} , \Pi_{T,2} , \ldots \Pi_{T,M}) \in \boldsymbol{R}^M$
- $\boldsymbol{P} \subseteq \boldsymbol{R}^M$ = the set of all cash flow vectors achievable by portfolios
  - A linear subspace -- may add portfolios, and scalar multiply
- $\boldsymbol{L} \subseteq \boldsymbol{P}$ = the set of all portfolios with cost = $\Pi_0 = 0$
  - A linear subspace of $\boldsymbol{P}$ -- may add zero cost portfolios, and scalar multiply
- There may be more than one set of weights that gives the same Π_T
- *Lemma*: If there is no arbitrage, then the cost, Π_0, is the same for any portfolio with the same output vector, Π_T.
  - Proof: otherwise, buy the cheap way (the cheaper set of weights) and sell the more expensive version (the other set of weights). That is an arbitrage.
- Thus, the cost is a linear function of Π_T
- Let n be a vector normal to $\boldsymbol{L}$ inside $\boldsymbol{P}$
- $\Pi_0 = C (n \cdot \Pi_T)$
  - Two linear functions that vanish together



“No Arbitrage” and “Risk Neutral Pricing”

- ***A*** = the set of portfolios with $\Pi_{T,j} \geq 0$ for all outcomes j = 1, …, M
- “No Arbitrage” means that ***L*** does not intersect ***A***, except at 0.
- In that case -- see figure -- n is inside ***A***.
- This means that the $n_j \geq 0$ for all outcomes j = 1, …, M.
- Define *risk neutral probabilities* $P_j = Cn_j$
  - $P_j \geq 0$ for all j, $P_1 + P_2 + \cdots + P_M = 1$ (through choice of C)

Π_0 = Portfolio cost

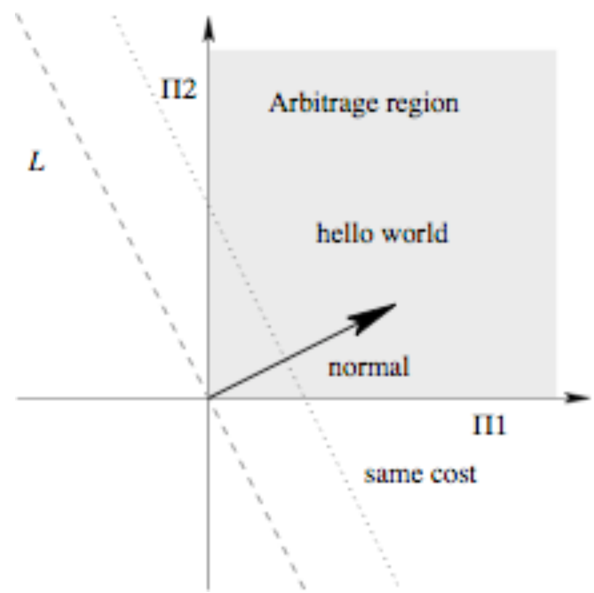
$= C\ (n\cdot\Pi_T\ )$

$= C_1\ (P_1\,\Pi_{T,1} + P_2\,\Pi_{T,2} + \cdots + P_M\,\Pi_{T,m})$

$= C_2\ E_P\,[\ \Pi_T\,]$

$$\Pi_0 = C_2\ E_{RN}[\ \Pi_T\,]$$

Price = discounted (C2<1) expected value



Complete market and replication

- A market is *complete* if $\boldsymbol{P} = \boldsymbol{R}^M$
- An *option* is a contract that pays U_j in state j at time T
- In a complete market, there is a portfolio, Π, with $\Pi_T = U$
- *Replication*: $\Pi_{T,j} = U_j$ for all states of the world, $j = 1,\ldots,M$
- In a complete market, any option can be replicated.
- In a complete market without arbitrage, the price of the replicating portfolio is uniquely determined by its payout structure, *U*
- If the option is traded at time 0, it is part of the market
- *Theorem*: assume that
  - The market with the option is arbitrage free
  - The market without the option is complete
- *Then*:
  - The option may be replicated
  - All replicating portfolios have the same price
  - That price must be the market price of the option
  - That price is the discounted expected payout in the risk neutral measure

$$\text{Price(option)} = C\, E_P[\text{ option payout }]$$

Complete market and replication, comments

- The risk neutral probabilities are determined by the complete market without the option -- they are the same for every extra option.
- If the market is complete, the risk neutral probabilities are uniquely determined by the market -- the direction of a normal to a hyperplane of dimension M-1 is unique.
- If the market is not complete, the normal direction within P is unique -- there are unique risk neutral probabilities for any option that can be replicated.
- If the option cannot be replicated, then there is a range of prices that do not lead to arbitrage.
- Real markets have *market frictions* that prevent arbitrarily small arbitrage transactions.
  - *Transaction costs*: portfolios with equivalent values at time T may have different costs at time 0.
  - *Limited liquidity*: the cost to buy n “shares” of asset i may not be proportional to n -- *move the market*.
- This material often is described differently, using linear programming.
- *Keith Lewis* told me it was easier to do it geometrically, as it is here.

Utility, risk neutral pricing

- Let X be an investment whose value in state j is X_j.
- Let Q_j be the *real world* probability of state j, possibly subjective.
- The real world expected value is

$$M = E_Q[\, X\,] = X_1 Q_1 + X_2 Q_2 + \cdots + X_M Q_M$$

- Fundamental axiom of finance: Price(X) ≤ M
- If variance(X) > 0, a *risk averse* investor has value(X) < M
- A risk neutral investor has value(X) = M
- The difference M - value(X) is the *risk premium* of X for that investor
- The difference M - price(X) is the *risk premium* of the market
- Risk premia depend on personal psychology and needs
- The market risk premium is determined by interactions between investors. It should be positive but is hard to predict quantitatively
- In this setup, it is hard to predict price(X) from first principles

- Risk neutral pricing says that there are risk neutral probabilities $P \neq Q$ so that price(X) = C $E_P[X]$, if X is an option payout in a complete market
- Since X can be replicated, value(X) is the same for every investor, and is equal to C $E_P[X]$.
- Can find prices of options without psychology.

Binary "one period" model

- The market has two instruments, *stock* and *cash* (also called *bond*)
- There are M = 2 states of the world "tomorrow", called "up" and "down"
- The value of "cash" today is 1
- The value of "cash" tomorrow is e^{rT}, r being the risk free rate
- The value of "stock" today is S_0
- The value of stock tomorrow is
  - u S_0 in state "up"
  - d S_0 in state "down"
  - Assume u > d
- This market is complete (check)

Risk neutral probabilities for the binary model

- With M = 2, the cost free portfolios form a one line
- W_s = weight of stock = a
- W_c = weight of cash = $-aS_0$ (to be cost free)
- Portfolio values at time T
  - $\Pi_{T,u} = aS_0(u - e^{rT})$
  - $\Pi_{T,d} = aS_0(d - e^{rT})$
  - Opposite sign (no arbitrage) if $d < e^{rT} < u$
- Normal: $(x,y) \Rightarrow (-y,x)$
- Normal to L: $(u - e^{rT}, d - e^{rT}) \Rightarrow (e^{rT} - d, u - e^{rT})$, both positive
- Normalize to get probabilities:
  - $n_u + n_d = u - d$
  - $n_u/(u - d) = p_u = (e^{rT} - d)/(u - d)$
  - $n_d/(u - d) = p_d = (u - e^{rT})/(u - d)$
  - Discount factor = e^{-rT}, otherwise risk free cash is an arbitrage
- If V is an option that pays (V_u, V_d), then the price of V today is

$$\text{price}(V) = e^{-rT}E_P[V_T] = e^{-rT}\big(V_u (e^{rT} - d) + V_d (u - e^{rT}) \big) /(u - d)$$

Binary model, Delta hedging

- A derivatives desk is asked to hold an option but does not want risk
- Short a *replicating portfolio,* Π*,* of stock and cash
- The total portfolio has zero value and zero risk.
- Make a profit from commissions.
- Replicating portfolio = Π = Δ Stock + C Cash,
- $\Pi_T = V_T$, both up and down
- $\Pi_0 = \Delta S_0 + C$
- $\Pi_{T,u} = \Delta u S_0 + e^{rT}C = V_u$
- $\Pi_{T,d} = \Delta d S_0 + e^{rT}C = V_d$
- Solve: $\Delta = (V_u - V_d) / (u S_0 - d S_0)$ = (change in V) / (change in S)
- $(V - \Delta S)_u = (V - \Delta S)_d$
- Δ hedged portfolio value at time T is not random, risk free
- Equivalent to cash, value known at time 0

Binomial multi-period model

- Times $0 = t_0, t_1, \ldots, t_N = T$, $t_k = k\delta t$
- Cash increases by $e^{r\delta t}$ between t_k and t_{k+1}
- S_0 = present spot price = known
- $S_{k+1} = uS_k$ or $S_{k+1} = dS_k$
- $S_1 = uS_0$, or $S_1 = dS_0$, as before
- $S_2 = u^2S_0$, or $S_2 = udS_0$, or $S_2 = d^2S_0$
- ud = du -- the binomial tree is *recombining* (diagram)
- N+1 possible values of $S_N = S_T$, 2^N if not recombining
- State j has j up steps and k - j down steps: $S_{kj} = u^j d^{k-j} S_0$
- European style option pays V_{Nj} at time t_N=T in state j
- V_{kj} = price/value of option at state j at time k
- V_{kj} is determined by $V_{k+1,j}$ and $V_{k+1,j+1}$ as before
- Work backwards:
  - Given all V_{Nj} values, calculate all $V_{N-1,j}$ values
  - Given all $V_{N-1,j}$ values, calculate all $V_{N-2,j}$ values
  - Eventually, reach V_0

Dynamic hedging, rebalancing in the binomial tree model

- At time t_k in state j, there is a hedge ratio $\Delta_{kj} = (V_{k+1,j+1} - V_{k+1,j})/S_{kj}(u\text{-}d)$
- This is how many shares of stock you own before you leave time t_k
- At time t_{k-1}, you probably had a different number of shares:
  - $\Delta_{k-1.j-1j}$ or $\Delta_{k-1,jj}$ neither one equal to Δ_{kj}
- When you arrive at time t_k, you have to replace the old number of shares with the correct number, Δ_{kj}. This is *rebalancing*.
- You pay for the new shares by spending your cash, this requires more borrowing if the cash position is negative.
- This is *dynamic hedging*, or
- *Dynamic replication*: $\Pi_T = V_T$ for any state at time T
- The dynamic hedging strategy produces a portfolio of stock and cash worth exactly V_{Tj}, if S_{Tj} is the state at time T.
- It is *self financing*. You generate the cash you need to buy stock. You keep the proceeds from selling stock.