Courant Institute of Mathematical Sciences

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Mathematics in Finance

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http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html

Hedging and pricing with a forward

1. Forward price for forward contract settled at time $T' \geq T$:

$$F_t \;=\; e^{r(T'-t)} S_t \;=\; e^{rT'} e^{-rt} S_t \;=\; C_{T'} e^{-rt} S_t \;.$$

2. (Up to a constant) F_t =exponentially discounted version of S_t
3. $F_0 = E_{RN}[F_t] = E_P[F_t]$, if $0 \leq t \leq T'$
4. If the possible values of F_T are F_u and F_d, then 3 implies

$$p_u \;=\; \frac{F_0 - F_d}{F_u - F_d}$$

5. Replicating portfolio with a forward and case: same

Random process

- $S_t = S(t) =$ price at time t
- S is the whole path, S_t is the value at time t
- $\mathcal{F}_t =$ all information available at time t, including $S_{t'}$ for $t' \leq t$ (see stochastic calculus)
- $u(s, t' \mid \mathcal{F}_t)$ (with $t' \geq t$) is the probability distribution of $S_{t'}$ given all information available at time t
- Discrete time: $0 = t_0 < t_1 < \cdots < t_n = T$
- Binomial tree model, homogeneous, $t_k = k\delta t$:
  - $S_{t_{k+1}} = uS_{t_k}$ with probability p_u
  - $S_{t_{k+1}} = dS_{t_k}$ with probability p_d
  - all steps independent

Martingale

- $\mathcal{F}_t$ is the *information available* at time t
- Assume nothing is forgotten: $\mathcal{F}_t \subseteq \mathcal{F}_{t'}$ if $t' > t$
- X_t is a *stochastic process* with respect to $\mathcal{F}_t$ if the value of X_t is determined by $\mathcal{F}_t$.
- Example: $\mathcal{F}_t$ is the prices of all listed stocks up to time t, X_t is an index.
- Example: X_t is the average of a stochastic process up to time t.
- M_t is a *martingale* if $E\left[M_{t'} \mid \mathcal{F}_t\right] = M_t$, for $t' \geq t$.
- Example, *simple random walk*: $a > 0 > b$, $ap_u + bp_d = 0$
  - $M_{t_{k+1}} = M_{t_k} + a$ with probability p_u
  - $M_{t_{k+1}} = M_{t_k} + b$ with probability p_d
  - All steps independent, $p_u + p_d = 1$
  - $\mathcal{F}_{t_k} =$ values of M_{t_j}, for all $j \leq k$, (including M_{t_k}, so $\mathcal{F}_{t_k}$ determines M_{t_k})

Risk neutral model worlds

- In the risk neutral measure, $F_t = e^{-rt} S_t$ is a martingale.
- $F_t = E_{RN}\left[F_{t'} \mid \mathcal{F}_t\right] \Longleftrightarrow E_{RN}\left[S_{t'} \mid \mathcal{F}_t\right] = e^{r(t'-t)} S_t$
- Lognormal model:
  $S_{t'} = S_t \exp\left\{(r - \sigma^2/2)(t' - t) + \sigma\sqrt{t' - t}\, Z\right\}$
- Binomial tree model, $t_k = k\delta t$, all steps independent
  - $S_{t_{k+1}} = uS_{t_k}$ with probability p_u
  - $S_{t_{k+1}} = dS_{t_k}$ with probability p_d
  - $up_u + dp_d = e^{r\delta t}$, $p_u + p_d = 1$
- $t' - t = \Delta t$ small, calibrate the binomial model so that $\Delta S = S_{t+\Delta t} - S_t$ have the same mean and variance as the lognormal model, conditional on $\mathcal{F}_t$.
- Price today of $V(S_T)$ at time T is $e^{-rT} E_{RN}\left[V(S_T)\right]$
  - Discounted expected value, in the *Risk neutral* model

The log process and log tree

- If S_t is a lognormal process, then $X_t = \log(S_t)$ is a "normal" process
  - $X_{t'} = X_t + \left(\mu - \frac{\sigma^2}{2}\right)(t' - t) + \sigma\sqrt{t' - t}\,Z$
  - $\Delta t = t' - t$, $\Delta X = X_{t'} - X_t$
  - $\Delta X \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right)$
  - $\Delta X^2 = O(\Delta t) \Longrightarrow \Delta X \sim O\left(\sqrt{\Delta t}\right) \gg \Delta t$

  The Ito calculus: $E\left[e^{\Delta X}\right] \approx 1 + E\left[\Delta X\right] + \frac{1}{2} E\left[\Delta X^2\right]$
- If S_t is a binomial tree process, then $X_t = \log(S_t)$ is a simple random walk
  - p_u and p_d do not change
  - $a = \log(u)$, $b = \log(d)$.

Continuous time limit

- Find an approximate description of geometric random walk (the binomial tree process) or ordinary random walk when δt is small
- Calculus vs. algebra
  - Algebra: simple foundations, complicated formulas

$$S(n) \;=\; \sum_{k=0}^{n} k^2 \;=\; \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \,.$$

  - Calculus: mathematically challenging foundations, simple formulas

$$\int_a^b f(x)dx \;=\; \lim_{\delta x \to 0} \sum_{x_k} f(x_k)\delta x \;\;, \quad x_k = a + k\delta x \,.$$

$$I(n) \;=\; \int_{x=0}^{n} x^2 dx \;=\; \frac{1}{3}n^3.$$

$$S(n) \;\approx\; I(n) \;\text{ for large } n.$$

Key 1: Central Limit Theorem

- $X_{k+1} = X_k + Y_k$, $\Pr(Y_k = a, b) = p_u, p_d$, $X_0 = 0$
- $X_n = \sum_{k=1}^{n} Y_k$, with the Y_k i.i.d. (independent, identically distributed)
- $\mu_Y = \overline{Y} = E[Y]$, $\sigma_Y^2 = \text{var}(Y) = E\left[\left(Y - \overline{Y}\right)^2\right]$.
- *Central Limit Theorem*: X_n is approximately Gaussian with mean $n\mu$ and variance $n\sigma^2$.
- Probability density of X_n is $f_n(x)$

$$f_n(x) \approx \frac{1}{\sqrt{2\pi n\sigma_Y^2}} e^{-(x-n\mu_Y)^2/(2n\sigma_Y^2)} .$$

- If $x = ja + (n-j)b$, then (in binomial simple random walk)

$$\Pr(X_n = x) = \frac{n(n-1)\cdots(n-j+1)}{j(j-1)\cdots 1} p_d^j \, p_u^{(n-j)} .$$

Key 2: Scaling

- $T = t_n = n\delta t$
- $\mu_Y = \delta t \mu$ ($\mu =$ growth rate).
  - $\mu_y =$ mean in a single time step, *scales like* δt
- $\sigma_Y^2 = \delta t \sigma^2$ ($\sigma =$ volatility)
  - $\sigma_y =$ standard deviation in a single time step, *scales like* $\sqrt{\delta t}$
- Then $E[X_T] = \mu T$ and $\text{var}(X_T) = \sigma^2 T$
- If $f(x, T)$ is the density of X_T, then

$$f(x, T) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-(x-\mu T)^2/(2\sigma^2 T)} .$$

- In simple random walk, have $\delta X = O(\sqrt{\delta t})$, could take
  - $a = -\sigma\sqrt{\delta t} + \mu\delta t$, $b = \sigma\sqrt{\delta t} + \mu\delta t$, $p_d = p_u = \frac{1}{2}$
  - $a = -\sigma\sqrt{\delta t}$, $b = \sigma\sqrt{\delta t}$,
    $p_d = \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\delta t}$, $p_u = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\delta t}$

Brownian motion

- $X_t = \sum_{t_k \le t} Y_k \sim$ normal mean μt, variance $\sigma^2 t$.
- $X_{t'} - X_t =$ increment between t and t'
  $\sim$ normal mean $(t' - t)\mu$, variance $(t' - t)\sigma^2$.
- Increments from disjoint intervals are independent
- $|X_{t'} - X_t| = O\left(\sqrt{t' - t}\right)$ for small increments
- Rough paths, not differentiable

Geometric random walk/binomial tree model

- $S_t = S_0 e^{X_t}$
- $d = e^b \approx 1 - b + \frac{1}{2} b^2$ (need the b^2, Ito)
- $u = e^a \approx 1 - b + \frac{1}{2} b^2$ (need the b^2, Ito)
- In the limit $\delta t \to 0$, S_t is *Geometric Brownian motion*
- $S_t = S_0 e^{X_t}$, where X_t is Brownian motion (gaussian).