Courant Institute of Mathematical Sciences

New York University

Mathematics in Finance

Derivative Securities, Fall 2009

Class 4

last corrected October 13 to fix page 12

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http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html

Ito calculus: keep small, neglect tiny

- Keep δt: $\displaystyle\sum_{t_k<T} a(t_k)\delta t \;\to\; \int_0^T a(t)dt \;\neq\; 0 \;,\quad \text{as } \delta t \to 0$
- Neglect higher powers:
  $$\sum_{t_k<T} a(t_k)\delta t^{3/2} \;\approx\; \left[\int_0^T a(t)dt\right]\delta t^{1/2} \;\to\; 0 \text{ as } \delta t \to 0$$
- Application: suppose $F_{k+1} = (1 + b(t_k)\delta t)\, F_k$
- Taylor series: $1 + \epsilon = e^{\epsilon - \frac{1}{2}\epsilon^2}$ (ϵ = small, ϵ^2 = tiny)
- $F_n \approx F_0 \exp\left(\sum_{t_k<T} b(t_k)\delta t\right) \exp\left(\sum_{t_k<T} \frac{1}{2} b(t_k)^2 \delta t^2\right)$
- $F_n \to F_0 \exp\left(\int_0^T b(t)dt\right)$ as $\delta t \to 0$

Geometric random walk/Brownian motion

- $S_{k+1} = X_k S_k$, $E\left[X_k\right] = 1 + \mu\delta t$, $\mathrm{var}\left(X_k^2\right) = \sigma^2\delta t$
- Write $X_k = 1 + Y_k$, with $E\left[Y_k\right] = \mu\delta t$
- $E\left[Y_k^2\right] = \sigma^2\delta t$ (small) $+ \mu^2\delta t^2$ (tiny) $\approx \sigma^2\delta t$.
- $S_n \approx S_0 \exp\left(\sum_{t_k<T} Y_k - \frac{1}{2}\sum_{t_k<T} Y_k^2\right)$
- Central limit theorem: $\sum_{t_k<T} Y_k \approx \mu T + \sigma W_T$
- Law of large numbers: $\sum_{t_k<T} Y_k^2 \approx \sigma^2 T$
- Altogether: $S_n \approx S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right)$
- The limit $\delta t \to 0$ depends only on $E\left[Y_k\right]$ and $\mathrm{var}\left(Y_k\right)$

Brownian motion

- $W(t)$ is a random function of t. W is a random path.
- Let $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$ be start and end times for disjoint intervals: $I_k = [s_k, t_k]$
- *Increments* of Brownian motion are $Y_k = W(t_k) - W(s_k)$
- It's standard ($\mu = 0$, $\sigma = 1$) Brownian motion if the following properties hold:
    1. W is a continuous function of t
    2. $W(0) = 0$
    3. The increments are independent Gaussian random variables with $\text{var}(Y_k) = t_k = s_k$.
- **Theorem**: Arithmetic (ordinary) random walk can be scaled and normalized so that it converges to Brownian motion as $\delta t \rightarrow 0$
- Therefore, a path whose increments are the sum of many smaller independent increments may be modeled as Brownian motion

Newton/Leibnitz/Reimann to Ito: Tiny becomes small

- $\delta W(t) = W(t+\delta t) - W(t)$ has
  - $E\left[\delta W^2\right] = \delta t$ and
  - $|\delta W| = O(\sqrt{\delta t})$
- Riemann sum, left ended
  $R_- = \sum_{t_k < T} f(t_k)\delta g(t_k) \longrightarrow \int_0^T f(t)dg(t)$
  as $\delta t \to 0$
- Right ended: $R_+ = \sum_{t_k < T} f(t_{k+1})\delta g(t_k)$
- If f and g are smooth, then $R_+ - R_-$ is tiny, because
  $R_+ - R_- = \sum_{t_k < T} \delta f(t_k)\delta g(t_k) = \sum_{t_k < T} O(\delta t^2) = O(\delta t T)$
- If $f = g =$ Brownian motion, then the difference is small:
  $R_+ - R_- = \sum_{t_k < T} \delta W(t_k)^2 \to T$ (law of large numbers)

Ito integral with respect to Brownian motion

- $\int_0^T F(t)dW(t) = \lim_{\delta t \to 0} F(t_k)\delta W(t_k)$

  left ended approximations only
- F may be random, but it must be *nonanticipating*
  - $F(t)$ is known at time t
  - Increments of W beyond t completely unknown
  - $E\left[F(t_k)\delta W(t_k)\right] = E\left[F(t_k)\right] E\left[\delta W(t_k)\right] = 0$
- $\int_0^T W(t)dW(t) = \frac{1}{2}\left(W(T)^2 - T\right)$
  - The expected value is zero, as it must be
  - If $W(t)$ were a smooth function of t, would not have $\frac{1}{2}T$
  - The right ended approximation gives $\frac{1}{2}\left(W(T)^2 + T\right)$, as it should

The Ito differential

- If $F(t)$ is a random function, then dF is defined by
  - $\int_{t_1}^{t_2} dF(t) = F(t_2) - F(t_1)$
- Get a formula for dF by calculating δF and keeping small terms and neglecting tiny terms.
- ϵ is *tiny* if
  - $|\epsilon| = O(\delta t^p)$ with $p > 1$, or
  - $|\epsilon| = O(\delta t)$ and $E[\epsilon] = 0$
- *Ito's rule*: if $|\epsilon| = O(\delta t)$, then $\epsilon = E[\epsilon] + \text{tiny}$.
- Example: $\delta(W^2)(t)$
  $= (W(t) + \delta W)^2 - W(t)^2 = 2W(t)\delta W(t) + (\delta W(t))^2$
  - $d\left(W(t)^2\right) = 2W(t)dW(t) + dt$
  - $W(T)^2 - W(0)^2 = \int_0^T d\left(W(t)^2\right) = 2\int_0^T W(t)dW(t) + \int_0^T dt$

Ito's lemma

- Suppose $u(w,t)$ is a smooth function of w and t.
  What is $du(W(t),t)$?

$$\begin{aligned}
\delta u &= u(W+\delta W, t+\delta t) - u(W,t) \\
&= \partial_w u\, \delta W + \frac{1}{2}\partial_w^2 u\, \delta W^2 + \partial_t u\, \delta t + \text{tiny terms} \\
&= \partial_w u\, \delta W + \frac{1}{2}\partial_w^2 u\, \delta t + \partial_t u\, \delta t + \text{tiny terms}
\end{aligned}$$

- Ito's lemma: $du(W,t) = u_w\, dW \; + \; \left(\frac{1}{2}u_{ww} + u_t\right)\, dt$
- Meaning #1: $u(W(T),T) - u(0,0) =$
  $\int_0^T u_w(W(t),t)dW(t) \; + \; \int_0^T \left(\frac{1}{2}u_{ww} + u_t\right) dt$
- Meaning #2: $dF = a(t)dt + b(t)dW$ if
  - $E\left[\delta F\right] = a(t)\delta t + \text{tiny}$
  - $E\left[(\delta F)^2\right] = b^2(t)\delta t + \text{tiny}$
  - $E\left[(\delta F)^4\right] = \text{tiny}$ (very technical – don't worry about it)

Ito differential equation

- An equation of the form $dX = a(X)dt + b(X)dW$.
- Seek a solution in the sense of meaning 2 above:
  - $E\left[\delta X\right] = a(t)\delta t + \text{tiny}$
  - $E\left[(\delta X)^2\right] = b^2(t)\delta t + \text{tiny}$
  - $E\left[(\delta X)^4\right] = \text{tiny}$
- Stock price process: $dS(t) = \mu S dt + \sigma S dW$
- Solution: $S(t) = S(0)\exp\left(\sigma W(t) + \left(\mu + \frac{1}{2}\sigma^2\right) t\right)$. Check it!

Black Scholes formula for a European put

- $S_T \sim S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T} Z\right)$ (same distribution)
- $I_A(s) =$ indicator function $= 1$ (if $s \in A$) or 0 (if $s \notin A$)
  Here, $A = \{S \leq K\}$
- $P =$ Put price
  $= e^{-rT} E\left[\, (K - S_T)_+ \,\right]$
  $= e^{-rT} E[\,(K - S_T)\, I_{S_T<K}(S_T)\,]$
  $= e^{-rT} K \Pr(\, S_T < K\,) \,+\, e^{-rT} E[\, S_T I_{S_T<K}(S_T)\,]$
- $S_T \leq K \iff Z \leq \frac{\ln(K/S_0) - \left(r - \sigma^2/2\right) T}{\sigma\sqrt{T}} = -d_2$
- $d_2 = \frac{\ln(S_0 e^{rT}/K) - \sigma^2 T/2}{\sigma\sqrt{T}}$, $\ln(S_0 e^{rT}/K) = \textit{moneyness}$
- $e^{-rT} K \Pr(S_T \leq K) = K e^{-rT} N(-d_2)$

BS, European put, cont...

- Need $e^{-rT}E[\,S_T I_{S_T<K}(S_T)\,]$, with $S_T = S_0 e^{rT} e^{-\sigma^2 T/s} e^{\sigma\sqrt{T}Z}$
- Get $S_0 e^{-\sigma^2 T/2} \int_{S_T \le K} e^{\sigma\sqrt{T}z} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$
- We just saw that $S_T < K \iff z \le -d_2$.
- Complete the square in the exponent:
  $\frac{-1}{2}\left(z^2 - 2z\sigma\sqrt{T}\right) = \frac{-1}{2}\left(\left(z - \sigma\sqrt{T}\right)^2 - \sigma^2 T\right)$
- So need (after cancellations):
  $S_0 \int_{z \le -d_2} e^{-(z-\sigma\sqrt{T})^2/2} \frac{dz}{\sqrt{2\pi}} = S_0 N(-d_2 - \sigma\sqrt{T})$
- $d_1 = d_2 + \sigma\sqrt{T}$,

BS, European put, final

- moneyness $= \ln(S_0 e^{rT}/K)$
  - dimensionless measure of the spot relative to the strike
  - moneyness $\to \infty$ (or $-\infty$) as $S_0 \to \infty$ (or 0) .

- $$d_2 = \frac{\text{moneyness} - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} = \frac{\text{moneyness}}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

- $$d_1 = d_2 + \sigma\sqrt{T} = \frac{\text{moneyness}}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

- $$P = e^{-rT}KN(-d_2) - S_0 N(-d_1)$$

- Check: $S_0 \to 0$ recovers forward contract price.

Black Scholes European call

- Do it all over again ... NO!
- Use put/call parity
  - In payouts (diagram): $\text{Call}(K, S_T) - \text{Put}(K, S_T) = S_T - K$
  - Prices today: $C(K, S_0) - P(K, S_0) = S_0 - e^{-rT}K$
- $C = P - Ke^{-rT} + S_0$, and
- $N(d) = 1 - N(-d)$
- Get:

$$C(K, S_0) \ = \ S_0 N(d_1) \ - \ e^{-rT} K N(d_2)$$

The Greeks

- Delta $= \Delta = \partial_S$(option value)
  - Used for Delta hedging: replicate using $\Delta_t S_t + \text{cash}_t$
- Gamma $= \Gamma = \partial_S^2$(option value)
  - Convexity: how nonlinear is your position?
  - How much rebalancing is needed?
- Vega $= \Lambda = \partial_\sigma$(option value)
  - Vol is unknown and/or has large unpredictable moves
  - The hedge depends on the vol.
  - A more robust hedge works well over a range of σ.
- Theta $= \Theta = \partial_t$(option value)
- Rho $= \rho = \partial_r$(option value)