Courant Institute of Mathematical Sciences

New York University

Mathematics in Finance

Derivative Securities, Fall 2009

Class 5

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http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec09/index.html

Review: discrete pricing and hedging

1. Model possible dynamics of the underlier:
   - $S \to uS$ or $S \to dS$
   - Probabilities not specified
2. Hedging/replication argument leads to $P(S_0, T)$ from the payout.
3. This algebra may be interpreted as e^{-rT} multiplying a weighted average of the payout values in various future states of the world.
4. This "weighted average" is interpreted as the "expected value" if the weights are the probabilities of the future states of the world.
5. These probabilities are called "risk neutral", because pricing something at its discounted expected value implies being "neutral" (indifferent) to risk.

Preview: how it goes in continuous time

1. Model dynamics of the underlier: $dS = \mu S dt + \sigma S dW$
   - Probabilities *are* specified specified
   - σ relevant (binary step size), μ irrelevant (binary probabilities)
2. Hedging/replication argument leads to the Black Scholes PDE
3. The solution of this PDE may be interpreted as the *value function* of a function of a random process:
   - $dS = rSdt + \sigma S dW$
4. Thus, the option price is the discounted expected value using a "risk neutral process"
5. The μ and r processes are related by a reweighting
   - Cameron Martin Girsanov formula
   - Processes related by reweighting are *equivalent*, but not the same.

Continuous time hedging

- $f(S(t),t) =$ market price of a traded option on underlier $S(t)$.
- $\Pi_t = \Delta_t S_t + C_t$ replicating portfolio of stock and cash.
- In time dt, $df = f_s dS + \frac{1}{2} f_{ss}(dS^2) + f_t dt$
- Ito: $dS = \mu S dt + \sigma S dW$, $(dS)^2 = \sigma^2 S^2 dt +$ tiny
- So: $df = \sigma S f_s dW + \mu S f_s dt + \frac{\sigma^2 S^2}{2} f_{ss} dt + f_t dt$
- Compute $d\Pi$ assuming $d\Delta_t = 0$: rebalance between trading periods.
- $d\Pi = \Delta S \sigma dW + \Delta S \mu dt + Crdt$
- Get $df = d\Pi$ only if $\Delta = f_s$
- Get the *Black Scholes Partial Differential Equation* (PDE)

$$\partial_t f \; + \; \frac{\sigma^2 s^2}{2} \partial_s^2 f \; + \; rs\partial_s f \; - \; rf \; = \; 0 \; .$$

Diffusions

- A *diffusion* process: $dX_t = a(X_t, t)dt + b(X_t, t)dW_t$
- $a(x, t) =$ drift coefficient
- $b(x, t) =$ noise coefficient
- In multidimensional diffusions, a is a vector and b is a matrix.
- $\delta X_t = X_{t+\delta t} - X(t)$, $\delta t > 0$
- $a(x, t)dt + \text{tiny} = E\left[\delta X_t \mid X_t = x\right]$
- $$\begin{aligned} b^2(x, t)dt + \text{tiny} &= \text{var}(\delta X_t \mid X_t = x) \\ &= E\left[(\delta X_t)^2 \mid X_t = x\right] + \text{tiny} \end{aligned}$$
- $E\left[|\delta X_t|\right] = O\left(\sqrt{\delta t}\right) >>$ drift motion
- Black Scholes model (Samuelson?) $dS = \mu S dt + \sigma S dW$
  - Drift $= a(s) = \mu s$
  - Noise $= b(s) = \sigma s$, $\sigma =$ volatility

Value functions

- $V(x)$ = final time payout
- $f(x,t) = E_{x,t}\left[V(X_T)\right] = E\left[V(X_T) \mid X_t = x\right]$
- Want only $f(x_0, 0)$, compute all $f(x,t)$ to get it
- e.g. X_t = Brownian motion, $V(x) = x^2$, $f(x,t) = x^2 + T - t$
- e.g. $dX = rXdt + \sigma XdW$, $V(s) = \mathbf{1}_{s \leq K}$, $f(x,t) = N(-d_2)$
- Related value functions
  - $f(x,t) = e^{-r(T-t)} E_{x,t}\left[V(X_T)\right]$ Discount
  - $f(x,t) = E_{x,t}\left[\int_t^T V(X_s)\, ds\right]$ Running gain (cost)
  - $f(x,t) = E_{x,t}\left[\exp\left(\int_t^T r(X_s) ds\right)\right]$ Interest rate models

Value function to PDE: the backward equation

- If $dX = a(X)dt + b(X)dW$, and $f(x,t) = E_{x,t}\left[V(X_T)\right]$:

$$\partial_t f + \frac{1}{2} b^2(x) \partial_x^2 f + a(x)\partial_x f \ = \ 0 \ . \quad \text{(backward equation)}$$

$$f(x,T) \ = \ V(x) \ . \qquad \text{(final conditions)}$$

- Derivation: tower property
- If $f(x,t) = e^{-r(T-t)} E_{x,t}\left[V(X_T)\right]$, then

$$\partial_t f + \frac{1}{2} b^2(x) \partial_x^2 f + a(x)\partial_x f \ + \ rf \ = \ 0 \ .$$

- If $f(x,t) = E_{x,t}\left[\int_t^T V(X_s)\, ds\right]$, then

$$\partial_t f + \frac{1}{2} b^2(x) \partial_x^2 f + a(x)\partial_x f \ + \ f \ = \ 0 \ .$$

Risk neutral process

- The stochastic process whose value function satisfies the PDE.